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THE HALL EFFECT IN BISMUTH WITH HIGH FREQUENCY CURRENTS.

By Alpheus W. Smith.

DES Coudres[1] has given a method for determining the Hall effect for alternating currents. The same current flows through the plate and the solenoid which produces the magnetic field in which the plate is placed. If the current in the plate and that in the solenoid are in the same phase, a direct electromotive force is developed at the Hall electrodes and this electromotive force can be measured in the usual way. Since the current in the plate and that producing the magnetic field must be in phase, the circuit must be free from iron and consequently only small magnetic fields can be realized. Observations made by Des Coudres on bismuth by this method indicate that the Hall effect for currents with a frequency of 50 or 700 per second is the same as for direct currents. The work of v. Traubenberg[2] on bismuth for currents with a frequency of 60 per second leads to the same conclusion. If the Hall effect is caused by the deflection of the electrons in the metal in some such way as the cathode rays are deflected by a transverse magnetic field, the electrons in the plate would always be deflected in the same way, since the magnetic field and the current in the plate alternate simultaneously. If for this or any other reason an appreciable time is required to build up the Hall electromotive force at the electrodes, the method of Des Coudres is not adapted to detect this fact. A method in which the direction of the magnetic field remains fixed and the current in the plate is alternating, so that the Hall electromotive force is also an alternating electromotive force, seems to give a more nearly correct measure of the Hall effect for alternating currents.

[1] Phys. Zeitschr., 2, p. 586, 1901.

[2] Ann. d. Phys., 17, p. 78, 1905.

Furthermore, Zahn[1] has shown that the method of Des Coudres can not be used for high frequency currents; for it is found that with such currents there are superposed on the Hall effect other effects which are similar to it but more intense. These effects are due to induced currents which, superposed on the primary current in the plate, cause temperature differences between the edges of the plate. These temperature differences set up thermal electromotive forces at the Hall electrodes which behave for the most part as the Hall electromotive forces. The presence of these pseudo-Hall effects make it impossible to determine the true Hall effect by this method.

It was the purpose of the following experiments to determine the Hall effect in bismuth for high as well as low frequency currents by a method which is free from the objections raised against the method of Des Coudres.

To measure the Hall electromotive force with alternating currents a Duddell thermal galvanometer was connected to the Hall electrodes A and B (Fig. 1). The heater H in the galvanometer was of very fine krupp-pin wire with a resistance of 3.48 ohms. The leads from the plate to the galvanometer were nearly all of No. 16 copper wire. It was necessary to solder short pieces of somewhat finer wires directly to the plate and then join the leads to these wires. The resistance of the leads amounted to 0.14 ohm. The deflection of the galvanometer is nearly proportional to the square of the current in the heater. The galvanometer was calibrated by sending a current from a storage cell of known electromotive force through the resistance R_1 and the heater. The resistance R_1 was adjusted until the deflection from the direct current was nearly equal to that from the alternating current. The square root of the mean square of the electromotive force between A and B is given by

$$e = \frac{R_h E \sqrt{d_1}}{R_1 \sqrt{d_2}},$$

where E is the electromotive force of the storage cell; R_h the resistance of the heater and leads; R_1 the resistance in series with the storage cell; d_1, the deflection of the galvanometer from the Hall electromotive force; and d_2, the corresponding deflection when the known direct current is sent through the heater. The Hall constant is then given by the usual equation,

$$e = R\frac{Hi}{d},$$

[1] Ann. d. Phys., 36, p. 553, 1911.

where H is the strength of the magnetic field in absolute units; i, the current in absolute units; and d, the thickness of the plate in centimeters.

The high frequency currents were obtained by joining two Duddell arcs X and Y in parallel as suggested by Nasmyth.[1] The currents thus obtained are much steadier than those obtained from a single Duddell arc. The bismuth plate WV in which the Hall effect was to be determined was placed in the secondary circuit which consisted of a self induction L and a capacity K joined to the arcs in the way indicated in Fig. 1. The high frequency currents were measured with a hot-wire ammeter A_2 of the form recently described by Fleming.[2] It consisted essentially of four fine copper wires in parallel, carrying the current to be measured. A thermal junction soldered to one of these wires was attached to a d'Arsonval galvanometer whose deflection gave a measure of the current flowing in the wire. The ammeter was previously calibrated with direct currents. By varying the capacity and self induction in the secondary circuit currents of different frequencies were obtained.

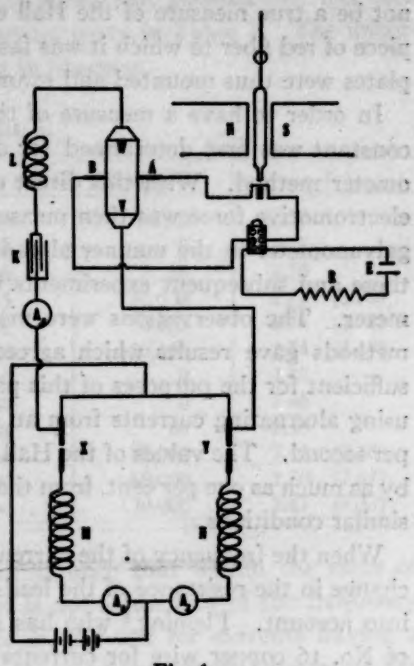

Fig. 1.

Mica condensers were used for the capacities and the frequencies were determined by tuning another circuit which contained a variable capacity and self induction so that it was in resonance with the circuit containing the plate. The frequencies were then calculated from Thomson's formula, $T = 2\pi\sqrt{LC}$.

The plates, which were made from Baker's pure bismuth, were cast in talc moulds in which there had been prepared a cavity of the form of the plate to be cast. From the middle of either side of the plate, which was 4 cm. long, 1.9 cm. wide, and 0.0685 cm. thick, there projected a narrow arm of bismuth 0.25 cm. wide and 1.7 cm. long. These arms served as Hall electrodes and to each of them was soldered a copper wire which led to the thermal galvanometer. To the ends of the plates

[1] PHYS. REV., 28, p. 459, 1909.

[2] Principles of Electric Wave Telegraphy and Telephony, p. 198.

were soldered strips of copper which served as primary electrodes by which the current came into the plate in such a way that the lines of flow were nearly parallel to the edges of the plate. By cutting the arms at their junctions with the plate it was possible to shift them along so as to make them lie very nearly on the same equipotential. Care must be taken to realize this condition or the deflection of the galvanometer will not be a true measure of the Hall effect. The plate was mounted on a piece of red fiber to which it was fastened by a coating of paraffin. Two plates were thus mounted and examined.

In order to have a measure of the accuracy of the method, the Hall constant was first determined for direct currents by the usual potentiometer method. With this direct current flowing in the plate, the Hall electromotive force was then measured by means of the Dudell thermal galvanometer in the manner already described. The magnetic field for these and subsequent experiments was 12,500 gausses per square centimeter. The observations were made at room temperature. The two methods gave results which agreed within one per cent., an accuracy sufficient for the purposes of this paper. The effect was then measured using alternating currents from an ordinary dynamo of about 60 cycles per second. The values of the Hall constant thus obtained did not differ by as much as one per cent. from that obtained with direct currents under similar conditions.

When the frequency of the currents becomes large the error due to the change in the resistance of the leads to the galvanometer must be taken into account. Fleming,[1] who has studied the change in the resistance of No. 16 copper wire for currents with a frequency of 1.06×10^6 per second, has found that for such frequencies the resistance is 6.62 times its value for direct currents. The value of the resistance for other frequencies can be calculated either from the formula given by Russell[2] or from that given by Kelvin.[3] When this change in the resistance of the leads is calculated it is found that it introduces an error of less than one per cent. for currents with a frequency of 30,000 per second; an error of about one and a half per cent. for currents with a frequency of 50,000 per second; and an error of about four per cent. for currents with a frequency of 120,000 per second. The self induction of the galvanometer leads would also introduce an error in the results. The self induction of the galvanometer circuit was calculated approximately from the formula given by Rosa and Grover.[4] By comparing the

[1] Proc. Phys. Soc. London, 33, p. 103, 1911.
[2] Proc. Phyc. Soc. London, 21, p. —, 1909.
[3] Jour. Inst. Elect. Eng., 18, p. 35 (1889).
[4] Bull. Bureau of Standards, 8, p. 152 (1912).

impedance of the galvanometer circuit with its ohmic resistance it is possible to estimate the magnitude of this error. It was thus found that when the frequency of the current was 120,000 per second, the observed value of the Hall constant would be two per cent. too low. For the other frequencies used here the error is less than one per cent. The influence of the capacity of the leads was small enough to be neglected. The necessary correction for the sources of error just pointed out has been made in the values of the Hall constants given in Table I. The uncorrected values have also been inserted in brackets.

TABLE I.

Plate.	Method.	Current in Amps.	Frequency.	R	
A	Potentiometer	3.58	0	2.77	
A	Duddell Galv.	3.58	0	2.75	
A	Duddell Galv.	3.60	60	2.75	
A	Duddell Galv.	3.45	30,000	2.73	(2.73)
A	Duddell Galv.	4.00	50,000	2.74	(2.70)
A	Duddell Galv.	3.65	120,000	2.74	(2.59)
B	Potentiometer	3.50	0	2.47	
B	Duddell Galv.	3.50	0	2.46	
B	Duddell Galv.	3.48	60	2.46	
B	Duddell Galv.	3.60	30,000	2.47	(2.46)
B	Duddell Galv.	3.80	50,000	2.50	(2.47)
B	Duddell Galv.	3.60	120,000	2.47	(2.33)

An inspection of this table shows at once that within the error of these observations the Hall constant is not a function of the frequency of the currents. Of these values of R, the one for currents having a frequency of 120,000 per second is the most uncertain because of the larger corrections which were used in calculating it. These corrections, which together amounted to about six per cent., were calculated with an accuracy such that the tabulated value of R does not differ by more than two per cent. from the true value. The error in the other values of R does not exceed one per cent.

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SOME ASPECTS OF THE DYNAMICS OF THE GYROSCOPE.

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BESIDE the algebraic simplicity due to selecting principal axes, which was probably Euler's single conscious aim, his dynamical equations for a rigid body present a notable peculiarity that he was less in a position to recognize completely. If the equations are written

$$M_1 = A\frac{d\omega_1}{dt} + (C - B)\omega_2\omega_3, \text{ etc.,} \tag{1}$$

the angular acceleration happens to be given accurately by the relation

$$\dot{\omega} = \frac{d\omega_1}{dt} + \frac{d\omega_2}{dt} + \frac{d\omega_3}{dt}, \tag{2}$$[1]

making it evident to inspection that the effects of applied force-moment are here segregated on an essentially kinematical basis. This results naturally from the trend of the original demonstration whose leading thought attaches to angular velocity and its changes;[2] and our compacter vector methods allow us to parallel and complete Euler's analysis in a few lines. For any element located by its radius-vector from the center of mass, the relative velocity, acceleration and force are

$$\boldsymbol{v} = [\boldsymbol{\omega r}];\ \dot{\boldsymbol{v}} = [\dot{\boldsymbol{\omega}}\boldsymbol{r}] + [\boldsymbol{\omega v}];\ dm\dot{\boldsymbol{v}} = dm[\dot{\boldsymbol{\omega}}\boldsymbol{r}] + dm[\boldsymbol{\omega v}]. \tag{3}$$

The corresponding total force-moment required is

$$\boldsymbol{M} = \int dm[\boldsymbol{r}\dot{\boldsymbol{v}}] = \int dm[\boldsymbol{r}[\dot{\boldsymbol{\omega}}\boldsymbol{r}]] + \int dm[\boldsymbol{r}[\boldsymbol{\omega v}]] = \boldsymbol{M}' + \boldsymbol{M}''. \tag{4}$$

Then simple routine transformation of the integrals shows for the components of $\boldsymbol{M}'$ and $\boldsymbol{M}''$ respectively,

$$A\frac{d\omega_1}{dt}, \text{ etc.; and } (C - B)\omega_2\omega_3, \text{ etc.;}$$

establishing in the two types of acceleration the basis of separation in the second member of eq. (1).[3]

[1] In the absence of a standardized notation, some reliance must be placed upon the context, where there are transitions back and forth between vectorial and algebraic statements, in order to prevent confusion. Keeping this in view, the intended sense of the equations here should become clear.

[2] Euler, Theorie der Bewegung (German trans. by Wolfers, 1853), pp. 431–443.

[3] This interpretation of the first term ($\boldsymbol{M}'$) is a corollary of a theorem announced by Minchin, Nature, Vol. 23, p. 62 (1881). The source of the second term ($\boldsymbol{M}''$) did not escape Euler himself (loc. cit., p. 323).

But since Euler's day it has been discovered how the idea really at the root of his original plan can be extended. In fact the propositions covering the more general use of so-called "moving axes" have been standard material for at least thirty years; it is therefore somewhat remarkable that they have been so neglected in actual application (except to the special case) until their more recent vector statement supplied a renewed stimulus.[1] It has become a familiar truth now, however, that with the usual assumption about symmetry (*i. e.*, $A = B$) in the gyroscope, we can retain Euler's scheme to the extent of continuing to use principal axes, without being limited by the condition underlying eq. (2). In other words the lines chosen to project upon may have any differential shift about the figure-axis, relative to the body itself. This plan offers an advantage in its more inclusive view, and does not deserve to be so nearly ignored as an alternative; thus the considerations that follow may be taken for a supplement to the "orthodox" presentation. At the same time, because they abandon the lines of kinematical description, and lean at some vital points rather toward directer dynamical statement, they foster a tendency to be encouraged.

We are to conceive the problem of the gyroscope in the usual way: A rotating rigid body having "universal-joint freedom" round a fixed point (O) is under control of weight-moment. With origin at this fixed point assume the following right-handed set of principal axes (possible when $A = B$): The figure-axis (C) with a perpendicular (B) to it in a vertical plane; and a third axis (A) normal to that plane. The resultant force-moment ($\boldsymbol{M}$) is then always being exerted about the instantaneous position of the line (A), which therefore coincides also with the vector derivative of the moment of momentum ($\boldsymbol{Mm}$). This implies for components and resultants; the latter being of necessity the actual "physical values":

(a) $\omega_1 + \omega_2 + \omega_3 = \omega$ (angular velocity),

(b) $A\omega_1 + B\omega_2 + C\omega_3 = \boldsymbol{Mm}$ (moment of momentum),

(c) $\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3 = \boldsymbol{M} = \dot{\boldsymbol{Mm}}$ (external force-moment).

Further we adopt the standard angular coördinates:

ϑ [measured from the downward vertical (Z) to the figure-axis],

ψ [specifying the azimuth of the plane (ZOC)],

ϕ [giving displacement relative to (ZOC) about (OC)].

When a progressive shift of axes is supposed a difference is introduced thereby between the two time-rates

[1] The main proposition did not find a place in the earlier editions of Routh's Dynamics; and though Klein and Sommerfeld (Theorie des Kreisels, p. 113) emphasize Hayward's exhaustive treatment of the matter (1854), still their equations are exclusively of the classic type (where Cartesian) subject to the condition of eq. (2).

$$\boldsymbol{M}\dot{\boldsymbol{m}} \text{ and } \frac{d}{dt}\{A\omega_1 + B\omega_2 + C\omega_3\} \equiv \frac{d'}{dt}\boldsymbol{M}\boldsymbol{m},$$

and gives occasion for a corrective relation that may be written

$$\boldsymbol{M}\dot{\boldsymbol{m}} = \frac{d'}{dt}\boldsymbol{M}\boldsymbol{m} + [\boldsymbol{u}\boldsymbol{M}\boldsymbol{m}]; \tag{5}$$

where without breaking away from principal axes we can have

$$\boldsymbol{u} = \dot{\psi} + \dot{\vartheta} + k\dot{\varphi}; \tag{6}$$

and Euler's equations reappear for $k = +1$. For application to present conditions $k = 0$; so that the difference of plan disappears when $\dot{\varphi} = 0$. Inserting the special value of $(\boldsymbol{u})$; also $A = B$, $\omega_1 = \dot{\vartheta}$; and anticipating the constant magnitude of $(C\omega_3)$, we find

$$\boldsymbol{M} = \boldsymbol{M}_1 = \frac{d}{dt}\{A(\dot{\vartheta} + \omega_2)\} + [\dot{\vartheta} + \dot{\psi}, \boldsymbol{M}\boldsymbol{m}]. \tag{7}$$

Without going into details we may remark that this expression is well adapted to a perspicuous grouping of important particular cases; the ordinary weight-pendulum, the spherical pendulum, the adjustments to regular precession. It is favorable besides that $\boldsymbol{M}_2$ and $\boldsymbol{M}_3$ are both zero always. Thus the Cartesian expansion of eq. (7) falls at once into the form

$$-mg\bar{r}\sin\vartheta = A\frac{d^2\vartheta}{dt^2} + \dot{\psi}\sin\vartheta C\omega_3 - \dot{\psi}\cos\vartheta A\dot{\psi}\sin\vartheta, \tag{8}$$

$$0 = A\frac{d\omega_2}{dt} + \dot{\psi}\cos\vartheta A\dot{\vartheta} - \dot{\vartheta}C\omega_3, \tag{9}$$

$$0 = C\frac{d\omega_3}{dt} + \dot{\vartheta}A\dot{\psi}\sin\vartheta - \dot{\psi}\sin\vartheta A\dot{\vartheta}. \tag{10}$$

These only elaborate the one central fact that no vector change in moment of momentum is ever being produced, except the increment along the instantaneous position of the A-axis. The scheme of projection follows up this sole positive controlling action by means of eq. (8); from which in the equivalent form,

$$-mg\bar{r}\sin\vartheta = \frac{d}{dt}(A\dot{\vartheta}) + [\dot{\psi}\boldsymbol{M}\boldsymbol{m}_{BC}], \tag{11}$$

we see at once that the external force-moment is absorbed completely into two effects: (1) Changed azimuth of moment of momentum found in the plane (BOC); (2) changed magnitude of the horizontal part $(A\dot{\vartheta})$. Any further changes therefore are in character internal readjustments; which must moreover harmonize with the absence of control expressed

by the first members of equations (9) and (10). The possibility of such automatic interchanges ceases to be puzzling when we remind ourselves that vector constancy is not in general consistent with constancy of projections upon lines subject to angular shift; indeed the vector product in expressions like eq. (5) marks and measures the need of external control when moment of momentum remains in constant relation to shifting axes. For instance, the clearest reading of eq. (9) understands the directional changes recorded in the last two terms to be made at the expense of the associated magnitude-change (or *vice versa*), simply because no external source of moment of momentum is available. The two familiar constant values characteristic of this problem are then not similar in their foundation; the moment of momentum about (OZ) is constant because that axis is permanently parallel to the weight, while the axis (OC) only happens to be unique among the shifting axes in its self-compensated process of gain and loss, owing to the condition $(A = B)$. The changes resulting from external cause and those entailed by internal constraint are evidently assembled in eq. (7), whose general plan in the second member is then seen to be a segregation according to the change in (1) magnitude and (2) direction, occurring in the moment of momentum. Yet plain and simple as the guiding thought here proves to be, it has been found difficult to dispel the inherited confusion in which such forms of statement remain involved, as a consequence of continuing to overlook the real intention of equations of motion; and their confessed limitations; while we shuffle the terms in them with indifference to everything but the formal mathematical validity of the results. The vital importance of clear perceptions on this issue is our justification for dwelling a little upon it.[1]

In their primary sense, equations of motion express a quantitative equivalence (equality) between a net total of external agency (cause) applied to a body, and the response (effect) in changes of that body's

[1] We can note the "physical inversion" implied in Euler's "Zentrifugalmoment" without serious abatement in our appreciation of his inventive genius and general soundness. Nor are we moved to find fault with Coriolis writing in 1835, when he formulates his particular discovery in terms of "Force centrifuge composée." But we must experience disappointment on encountering in the best modern book on the gyroscope these survivals from the early tentative period still masquerading as pseudo-causes; and imbedded incongruously enough in a treatment whose main lines follow Kelvin's lucid dynamical thought. It is psychologically interesting to observe the persistent perversion due to so slight a turn as writing eq. (1) first

$$M_1 = A\frac{d\omega_1}{dt} - (B - C)\omega_2\omega_3,$$

and then

$$M_1 + (B - C)\omega_2\omega_3 = A\frac{d\omega_1}{dt}.$$

state of motion. Conceived in such terms, this "equality of cause and effect" introduces no metaphysical obscurity; nor is it an obstacle to flexible adaptation in the later stage of calculation; but when accurately considered, it reveals two cardinal limitations in the scope of the equations. First the equality subsists between two summations which are only equivalent in their net totals and not identical or interchangeable piecemeal; as eq. (4) or eq. (8) exemplifies. And secondly the constituent items of each summation can be varied indefinitely without disturbing those totals. The reduction to resultant force and force-moment being founded essentially on the work-equation, we recognize readily the substitution of them for the actual distribution of force as being on a basis of limited equivalence. And this is matched in the second member by a wide range in the admissible plans of describing the effects. Both sides of the equation of motion being in these respects artificialized for convenience, we must not seek to extract from either in general a "one to one correspondence" with all details of the physical conditions. The occurrence of a term in the description implies nothing conclusive about a corresponding physical action; and every suggestion becomes illusory when we obliterate the distinctions through original entry under the two rubrics of cause and effect. So the few forms found helpful toward a natural interpretation become conspicuously prominent. The terms of eq. (4) accord with one most fundamental idea: That a rigid body distributes automatically the applied external force, assigning its quota to each mass-element, calculable in terms of local acceleration. The principle illustrated in application by eq. (8) covers the direct expression of force-moment through changes in moment of momentum. But the process of superposition must not be pushed beyond the limits of its validity. This question regarding eq. (10), for example, is empty of content: Do the last two terms assert that these specific elements of force-moment are present; or is this merely a roundabout method of effectively denying the presence of any such elements? Let us accept explicitly these limitations as they may have bearing upon the matters that follow, where an attempt is made at adding dynamical significance to some results that are ordinarily left in the stage of algebraic proof.

Return to eq. (8) and recast it into the form

$$A\frac{d^2\vartheta}{dt^2} = \sin\vartheta\{-mg\bar{r} - C\omega_3\dot{\psi} + A\dot{\psi}^2\cos\vartheta\}. \qquad (12)$$

As we may see from the discussion of eq. (11), a zero value of this first member entails complete absorption of force-moment in causing purely directional changes of moment of momentum; and then adding the condi-

tion $\dot{\vartheta} = 0$ steadies the motion into regular precession, with the available force-moment exactly adequate to the effects then necessary.[1] This doubly specialized adjustment is seen from eq. (12) to be in every case possible at $\vartheta = 0$ or π for any finite factors within the parenthesis, and commonsense confirms the conclusion; since, though no weight-moment is available, neither is any then necessary, because the vertical (OZ) has become both a principal axis and the axis of rotation. But there remain the adjustments (in a fuller sense) determined by a zero value of the parenthesis itself, with their possible assignment to real and imaginary regions according to the associated parameters in $0 = f(\dot{\psi}, \vartheta)$. Where an imaginary region occurs it is found to separate the two types of real solution, which coalesce when it disappears; so that considerations developed previously for the conical pendulum[2] apply in parallel here. But the less restricted problem of the gyroscope outruns that parallelism, notably in allowing a fast or a slow precession; two adjustment-values of $\dot{\psi}$ corresponding to elements unvaried otherwise. We shall concern ourselves next with the dynamics underlying this possibility, whose equation of condition has been presented in the two forms:[3]

$$\dot{\psi} = \frac{C\dot{\varphi} \pm \sqrt{(C\dot{\varphi})^2 + 4mg\bar{r}(A - C)\cos\vartheta}}{2(A - C)\cos\vartheta}$$

and

$$\dot{\psi} = \frac{C\omega_3 \pm \sqrt{(C\omega_3)^2 + 4mg\bar{r}A\cos\vartheta}}{2A\cos\vartheta}. \qquad (13)$$

Ordinarily the comparison of the roots $(\dot{\psi}_1, \dot{\psi}_2)$ is best attached to the second form, on account of the constant (and therefore common) value of $(C\omega_3)$. The other seems to seek a mathematical purity in being formally explicit, whereas $C\omega_3 = C(\dot{\varphi} + \dot{\psi}\cos\vartheta)$. But this advantage is illusory in practice, if we are obliged to make allowance for variation in $(\dot{\varphi})$ in order to collate the two pairs of roots under most natural conditions. It will fit our present purpose, however, to note the emphasis of the first form in so far as it follows the superposition

$$\omega = \dot{\psi} + \dot{\varphi} + \dot{\vartheta}, \qquad (14)$$

and the building up of moment of momentum by stages that match.

[1] Eq. (11) follows the thought more closely, if we suppress the first right-hand term. Such "equations of adjustment" (type $Q = R$) are different from "equations of equilibrium" (type $Q + R = 0$), however slight the mathematical barrier between them. This is sometimes overlooked; zero = parenthesis of eq. (12) being classed erroneously as a "condition of equilibrium."

[2] Slate, PHYS. REV., Vol. 21, p. 166.

[3] The first is apparently peculiar to Klein and Sommerfeld (loc. cit., p. 178) among the prominent writers on the subject.

If positions of the figure-axis on the boundary of a quadrant are excluded, the vertical is not a principal axis; and consequently at the first stage the moment of momentum comprises a horizontal part ($H = \dot{\psi}(A - C)\sin\vartheta\cos\vartheta$) in the plane ($ZOC$) as well as the vertical part ($V = I_z\dot{\psi}$). Stopping at precession-adjustment, the scheme is completed by adding the component round the figure-axis ($C\dot{\varphi}$), and the alternative roots going with a common value of ($C\omega_3$) must in view of eq. (11) satisfy the relation

$$\begin{aligned} -mgr\sin\vartheta_1 &= [\dot{\psi}_1, I_z\dot{\psi}_1 + \boldsymbol{H}_1 + C\dot{\varphi}_1] = [\dot{\psi}_2, I_z\dot{\psi}_2 + \boldsymbol{H}_2 + C\dot{\varphi}_2]; \\ &= [\dot{\psi}_1, \boldsymbol{H}_1 + C\dot{\varphi}_1] = [\dot{\psi}_2, \boldsymbol{H}_2 + C\dot{\varphi}_2]. \end{aligned} \tag{15}$$

On the mathematical side, therefore, this double possibility of adjustment implies merely a numerical equivalence of vector products consistent with certain variations in the factors; though our physical analysis may profitably go one step further. A single type of case will illustrate the point of view sufficiently; it is not necessary to exhaust the algebraic combinations; so let us choose the factors under the radical in eq. (13) all positive, making the roots opposite in sign. Assume $\dot{\psi}_1 > 0$; $H_1 > 0$; beside $C\omega_3 > 0$. This gives $\dot{\psi}_2 < 0$; $H_2 < 0$; $|\dot{\psi}_1| > |\dot{\psi}_2|$; $|H_1| > |H_2|$; $\dot{\varphi}_2 > \dot{\varphi}_1$; and both values positive in the last inequality agrees with ratios of magnitude found in actual problems. Then the conventions adopted already, with the standard rule governing vector products, lead for the two roots to the effective signs:

$$-mg\bar{r}\sin\vartheta_1 = \begin{cases} -[\dot{\psi}_1\boldsymbol{H}_1] + [\dot{\psi}_1, C\dot{\varphi}_1] \\ -[\dot{\psi}_2\boldsymbol{H}_2] - [\dot{\psi}_2, C\dot{\varphi}_2]. \end{cases} \tag{16}$$

In connection with the relations of absolute magnitude indicated above, eq. (16) exhibits clearly how the same total force-moment can be partitioned differently between constituents at the two adjustments; so this at once enforces the ideas immediately preceding eq. (12) and also justifies an interpretation, valuable notwithstanding some limitations there set forth, which takes its place alongside those attached to eq. (4) and eq. (7). What reappears everywhere is the subjective core of the descriptive process; reconciling simultaneous constancy and change (by superposition), and treating different sets of components, each in its turn, as equally real content of the resultant.

Again we make eq. (8) the starting-point, this time in order to observe in its first member that the total work is being done continually about the A-axis, while its second member plainly involves the exhibition of resulting kinetic energy elsewhere than in connection with the ϑ-coordinate. The fact that work can be thus "transferred between co-

ordinates," even though they are perpendicular, is at the root of the dynamic stability which becomes so prominent in the gyroscope; and the completeness or the rapidity of such transfer enables us in a way to measure that form of stability or to set the limits of its range. The expression for the total kinetic energy is

$$E = \frac{A}{2}(\dot{\vartheta}^2 + \omega_2^2) + C\frac{\omega_3^2}{2}; \tag{17}$$

and the last term being constant, the variations or interchanges consequent upon work done are confined to the two other terms. Now referring to eq. (9) it is apparent that the initiative (so to speak) centers in the component that is in the line of the external force-moment. So long as $\dot{\vartheta} = 0$ no change can occur in (ω_2); but the vanishing of (ω_2, ω_3) separately or simultaneously does not prevent changes in $(\dot{\vartheta})$. This discriminates effectively between stable equilibrium and the "stability" here; the latter depends vitally upon the entrance of (at least) incipient displacement, and is in this respect similar to the "stoppage of motion" (necessarily incomplete) by eddy currents. This feature of gyroscopic mechanisms is always emphasized—that their efficiency is nullified by removing the essential "degree of freedom."

But the influence of the variable components $(\dot{\vartheta}, \omega_2)$ is nevertheless mutual; for we see from eq. (12) that the first member may be made positive or negative, with weight-moment given, by due assignment of the other elements. And in the next place it is clear that the absorption of kinetic energy *away from* the ϑ-coördinate will be more rapid; and therefore completed within smaller displacement if continued; in proportion as that first member has large absolute magnitude, while the first time-rate of (ϑ) and the second are opposite in sign. Hence a change of sign in the second time-rate is a critical boundary between conditions favorable and those unfavorable to that process of transfer (*i. e.*, to stability); and the critical equation marking the limit of favorable inequalities is

$$0 = -mg\bar{r}\sin\vartheta - C\omega_3\dot{\psi}\sin\vartheta + A\dot{\psi}^2\cos\vartheta\sin\vartheta. \tag{18}$$

By way of illustration, apply this line of thought to a case of central importance; the examination of stability for a gyroscope with figure-axis pointing vertically upward, and passing through that position $(\vartheta = -\pi)$ with angular velocity $\dot{\vartheta} > 0$. Then if this angular velocity is not to increase, nor even to persist, but is to be vigorously "nipped in the bud," $d^2\vartheta/dt^2$, though accurately zero at $\vartheta = -\pi$, must become (strongly) negative at departure from the vertical, and remain negative throughout

a sufficient interval. For $\vartheta = -\pi + \Delta\vartheta$, this leads obviously to the inequality, if we can assume $\cos\vartheta = -1$,

$$0 > +mg\bar{r} + C\omega_3\dot{\psi} + A\dot{\psi}^2. \tag{19}$$

But at coincidence of figure-axis and upward vertical there is a well-known relation, taking the form for our conventions,

$$\dot{\psi} = -\frac{C\omega_3}{2A}; \tag{20}$$

and using this we may simplify inequality (19) into either

$$(C\omega_3)^2 > 4Amg\bar{r}; \quad \text{or} \quad A\dot{\psi}^2 > mg\bar{r}. \tag{21}$$

In rough summary the lesson here amounts to this: The demands of the directional changes in moment of momentum must develop more than rapidly enough to monopolize the available force-moment as the figure-axis leaves the vertical. Of course the same method can be adapted to less special values of (ϑ); and it can be shown in detail that the energy-relations conform to the requirements outlined.[1] The present purpose is attained in suggesting these aspects of the phenomena. But as a closing word it may be well to remark that the "transfer of work" here spoken of is far from being peculiar to the gyroscope; and it has nothing fundamentally mysterious about it, being introduced as a possibility whenever the resultant force changes direction; as the resultant force-moment does here. It is instructive to consider in parallel the trajectory (weight being regarded as a constant vector) and a planetary orbit at the perihelion. The latter shows the entire kinetic energy gained since passage through the aphelion in connection with a direction instantaneously perpendicular to the resultant force. And with any central force, too, a certain constancy of moment of momentum is dominant, as it is in the gyroscope.

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[1] The point of view alone is claimed as novel. For the results, cf. Klein and Sommerfeld (loc. cit., pp. 249, 321) where they are extracted from the analytic discussion of the graph of a cubic equation.

THE ELASTIC PROPERTIES OF BISMUTH WIRES.[1]

By J. E. Harris.

ALMOST striking deviation from the laws of elasticity was first observed a few years ago by Guthe[2] and investigated by him[3] and by Sieg[4] in wires of platinum-iridium alloys. They found that the period of torsional vibration for these wires was by no means constant, but that it decreased rapidly with the amplitude of vibration. The greatest percentage of decrease was found in the smaller amplitudes, the periods seeming to approach a maximum constant value at the larger amplitudes. The logarithmic decrement, starting with the large amplitude, at first increased slowly to a maximum value and then decreased much more rapidly as the amplitude of vibration decreased. The previous treatment of the wire seemed to influence its behavior greatly. After allowing the wire to rest for a long time, or after annealing the wire by heating it to a red heat by means of an electric current, the peculiar properties were not nearly so pronounced as when the wire had just previously been vibrated torsionally for some time. It was also observed that when the wire was kept in vibration for some time at a constant amplitude, the period gradually increased until it finally approached a maximum constant value.

The fact that the previous history and treatment of the wire seemed to influence its behavior to such an extent led to the assumption that the effect was in some way due to the molecular structure of the wire. Since the platinum-iridium alloys are very brittle, it was suggested that perhaps other wires having this same quality of brittleness might show something of the same properties. For this reason, it was decided to investigate the behavior of bismuth wires. This was done and as has been announced by Guthe[5] and the author, the bismuth wires did show the same large decrease in period and logarithmic decrement with decrease in amplitude.

This paper gives the results of the investigations that have been carried on in the past year with bismuth wires.[6]

[1] This investigation was carried out with the aid of a grant from the Elizabeth Thompson Science fund.

[2] K. E. Guthe, Proc. Iowa Academy of Science, 15, p. 147, 1908.

[3] K. E. Guthe and L. P. Sieg, Physical Review, Vol. 30, No. 4, 1910.

[4] L. P. Sieg, Physical Review, Vol. 31, No. 4, 1910.

[5] Physical Review, Vol. 32, p. 228.

[6] The bismuth wires used in these investigations were obtained from Hartman & Braun. They were made from pure, electrolytically deposited bismuth.

II. APPARATUS.

The apparatus was similar in most respects to that described by Guthe and Sieg[1] in their investigations with the platinum iridium wires. A large circular scale was used of such dimensions that each degree of amplitude measured about 2 cm. on the circumference of the scale. Each end of the wire was soldered into a brass cylinder. For the purpose of soldering the wires, it was found necessary to mix some bismuth with the solder in order to lower its melting point to a temperature below that for bismuth, the melting point of pure bismuth itself being lower than that of the solder. The upper cylinder was clamped in a device shown in the accompanying diagram. The sketch gives a view at right angles to the axis of the wire. The cylinder to which the upper end of the wire is soldered is held in place by the set-screw *A*. The flat piece of iron *BCDE* carrying the cylinder is free to rotate about the axis of the wire as a center of rotation. By pushing this first one way and then the other, the wire could be made to vibrate. By means of the set-screws *F* and *G*, the angle through which the wire was twisted could be regulated at will. By setting these screws, the wire could also be made, if desired, to vibrate at constant amplitude. The part *HIJK* is a fixed piece of iron and has at one end of it the circular scale *mn*. By means of this scale, the angle through which the wire is being twisted can be determined.

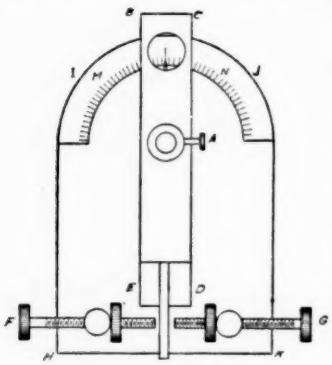

Fig. A.

The cylinder attached to the lower end of the wire was turned in such a way as to leave at its lower end a flat disk 5.58 cm. in diameter. Over the stem part could be slipped cylinders of various masses and diameters, so that the mass and moment of inertia of the vibrating system could be varied as desired. A very thin piece of plane parallel glass, silvered on both sides and fastened vertically to the lower surface of the vibrating disk, served for the reflection of light from an arc lamp to the scale. The lower cylinder was hung in a large beaker made of clear glass, the top of the beaker being covered with a piece of cardboard having a hole in the center through which the wire could pass. This arrangement was made necessary, especially in those observations in which a light suspended weight was used, because it was found that the air currents in the room interfered materially with the observations. The beaker was large

[1] Loc. cit.

enough so as not to introduce any damping effect. A cross-wire placed in a beam of light from an arc lamp was focused on the scale after reflection by the mirror by means of a lens.

The periods of vibration were obtained with the aid of a Morse recorder, one of the electromagnets of the recorder being connected to a relay which in turn was connected to a clock beating seconds, the other electromagnet being connected to the apparatus used for determining the time of the passage of the cross-wire through the point of rest. By measuring the distance between the marks on the tape, an accurate determination of the time of each passage of the cross-wire through the point of rest could be obtained.

In the earlier observations, instead of determining the time of passage of the cross wire through the point of rest by means of a tap-key, a selenium cell was placed in the position occupied by the spot of light reflected by the mirror when the vibrating system was at rest. The cell, which was very sensitive, having a resistance of 198,000 ohms in the dark and only 8,000 ohms in direct sunlight, was placed in series with a two-thousand ohm relay and both connected in a 220-volt circuit. The relay was so adjusted that when the light from the mirror passed over the cell in series with the relay, the decreased resistance of the cell would permit enough current to pass to close the relay, which, in turn closed a circuit containing one of the electromagnets of the recorder, thus giving a mark on the tape. For the larger amplitudes, the entire surface of the cell (about $5 \times 2\frac{1}{2}$ cm.) was exposed to the action of the light. However, as the amplitudes of vibration decreased, the spot of light remained for a continually lengthening period of time on the surface of the cell, so that the mark on the tape could no longer be given at the time of the passage of the spot of light over the middle of the cell but before it reached the middle. For this reason, two pieces of card-board were so arranged that they could be closed in from the ends, thus leaving a continually decreasing portion of the cell exposed to the action of the light. It was found that the slit could be narrowed down in this manner until it was considerably less than a centimeter in width. Even this was too wide for amplitudes less than 20° so that when the amplitude had decreased to this point, the records of the passages of light were obtained by means of a tap-key. Unfortunately, due to the passage of too much current through the cell in one of the observations, the sensitiveness was decreased to such an extent that it could no longer be used, and a tap-key had to be substituted in the observations taken thereafter.

III. Method of Procedure.

The object of the first experiment attempted was to determine the way in which the wire behaved as the initial amplitudes of successive observations were gradually increased. For this purpose, a series of experiments was carried on with two different wires, one of .25 mm. diameter and the other .5 mm. diameter, in each case beginning with an observation in which the initial amplitude was small and increasing the amplitude in each succeeding observation. The amplitudes were increased in each case until the elastic limit of the wire was very nearly reached. For all the experiments, a smooth period amplitude curve could be obtained, but when the curves were compared, no definite relation could be obtained. The curves for experiments, in which the initial amplitude was large, would sometimes be above and sometimes below those in which the initial amplitude was small. This was probably due to the different treatment of the wire on different occasions. It was necessary to adjust the relay in series with the selenium cell as the wire was being vibrated in order to bring it up to the desired initial amplitude and sometimes this adjustment was difficult to bring about, especially if the light from the arc happened to be a little variable. For this reason, it was necessary to keep the wire in vibration much longer in some experiments than in others.

From these results, it became evident that some method of procedure. would have to be found by which more definite results could be obtained In Sieg's[1] work, the method was tried of keeping the wire in constant vibration for some time at the maximum amplitude at which it was desired to begin the experiment and then taking a series of observations as the wire came to rest. It was found that if this experiment was repeated at once without the preliminary treatment, the period amplitude curves for the two sets of readings coincided exactly, indicating that the wire, for the time being at least, had been put in a definite state.

A series of experiments were performed to see if in this respect the bismuth wire behaved like the platinum-iridium wire.

1. *Comparison of Two Series of Observations taken One Immediately After the Other.*—In order to determine definitely the effect of the previous treatment of the wire, two series of observations were taken one immediately after the other. In this experiment, a wire was used that had been used in the experiments mentioned above. The wire had, however, been allowed to rest several days before these experiments were performed. The wire was made to vibrate in each case through an initial amplitude of about 85° and the observations taken as the system

[1] Loc. cit.

came to rest. In this and all other experiments with the bismuth wires, we are limited to small amplitudes compared to those used in the platinum-iridium wires, because of the low elastic limit of the wire. By period in this and in subsequent experiments is meant the period of half a complete vibration. The results for the two experiments are given in Table I., *A* being for the first and *B* for the second series of readings.

TABLE I.

Length of wire, 106 cm. Diam., .5 mm. Moment of inertia, 626 g. $\overline{\text{cm}}^2$. Temp., 23°. Initial amplitude, 86°.3.

No. of Swing.	*A*		*B*	
	Period.	Amp.	Period.	Amp.
5	3.223	73.4	3.224	74.0
10	3.219	63.1	3.221	63.3
15	3.216	54.0	3.217	54.3
20	3.213	46.7	3.215	45.8
25	3.211	40.0	3.213	40.5
30	3.209	34.6	3.210	34.7
35	3.206	30.0	3.207	30.2
40	3.203	26.1	3.205	26.3
45	3.201	22.8	3.203	23.0
50	3.199	19.8	3.201	20.0
55	3.197	17.4	3.200	17.5
60	3.195	15.3	3.198	15.4
65	3.194	13.5	3.196	13.5
70	3.192	11.9	3.194	11.8
75	3.191	10.6	3.193	10.5
80	3.189	9.3	3.192	9.2
85	3.188	8.4	3.190	8.3
90	3.187	7.5	3.189	7.4
95	3.187	6.7	3.188	6.6
100	3.186	6.1	3.187	5.7
105	3.185	5.4	3.187	5.1
110	3.185	4.9	3.186	4.7
115	3.184	4.4	3.185	4.3
120	3.183	3.8	3.185	3.7

The results of this experiment are shown graphically in Fig. 1, the amplitudes being plotted as abscissæ and the periods as ordinates. That the first experiment has had a marked effect on the second is evident from the position of the two curves. The curve for the second experiment lies above that for the first. It will be seen that the second curve in shape is very similar to that of the first. It seems that the effect of the first experiment on the second was merely to displace the curve upward. This effect is due to the so-called "elastic fatigue" in the wire.

2. *Behavior of the Wire when Vibrated at Constant Amplitude.*—In this experiment, several series of observations were taken at different amplitudes to determine whether or not the period remained constant when

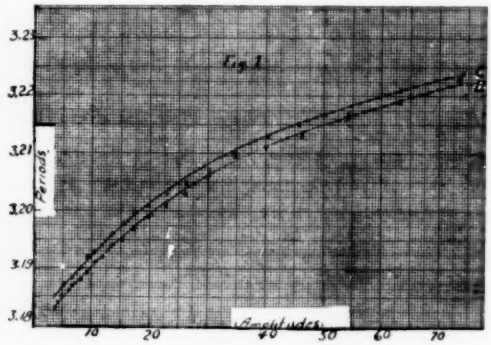

Fig. 1.

the wire was kept vibrating at constant amplitudes. By adjusting the set screws *F* and *G* (see diagram) and moving the lever from one set screw to the other at the end of each swing, the wire could be kept vibrating through any desired amplitude and this amplitude kept constant

TABLE II.

Length of wire = 106 cm. Diam. = .5 mm. Moment of inertia = 626 g. $\overline{\text{cm}}^2$.

A. Amplitude = 65°.2.

Coincidence.	Intervals.	No. of Vibrations.	Period.
21.21			
53.42	32.21	10	3.221
108.29	87.08	27	3.225
205.13	183.92	57	3.227
After 20 minutes.			
1.56			
85.57	84.01	26	3.231
130.78	129.22	40	3.231
260.07	258.51	80	3.231
B. Amplitude = 93°.5.			
37.70			
79.63	41.93	13	3.225
115.12	77.42	24	3.226
192.61	154.91	48	3.227
After 20 minutes.			
1.54			
62.96	61.42	19	3.233
224.65	223.11	59	3.233
337.82	336.28	104	3.233

within one or two degrees. Only two of the experiments performed are given here for the reason that all of them gave practically the same results. Two series of observations were taken in each case, the first just after the amplitude desired had been reached, and the second, after the wire had been kept in vibration for 20 minutes. The results are shown in Table II.

It will be seen from the above experiments, that the period of vibration increases when the system is made to vibrate at a constant amplitude; that the period of vibration very rapidly approaches a maximum value; and that this maximum value has apparently been reached after an interval of 20 minutes. To make sure of this latter point, the wire was vibrated for twenty minutes, an observation taken and then the wire vibrated for an hour and another observation taken. The period obtained was the same as that obtained after the twenty-minute interval. That a part of the adjustment of the wire takes place while the wire is being vibrated to bring it up to the maximum amplitude is shown by the above tables. In observation *A*, where the wire was brought to its maximum amplitude by larger steps than in *B*, we have a larger variation in the period in the first 50 vibrations than in the corresponding interval in the second observation.

3. *Effect of Continued Vibrations on the Period Amplitude Curve.*—To show this effect, a wire that had not been used before was taken and after allowing it to hang in place for a day, three series of observations were

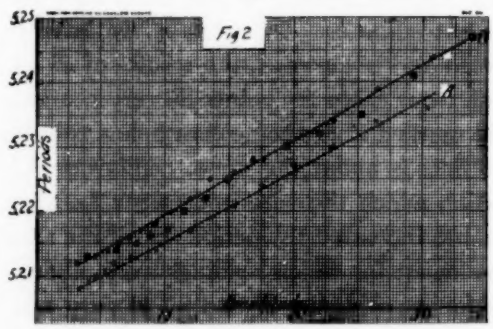

Fig. 2.

taken. In the first, the wire was brought up to the desired amplitude with as few preliminary swings as possible and a series of observations taken as the wire came to rest. In the second experiment, the wire was kept vibrating for a period of about thirty minutes at the same amplitude as that at which the first experiment was begun before taking the series of observations. The third experiment consisted in repeating the first experiment immediately after the second had been completed. The results appear in Table III.

TABLE III.

Length of wire, 103.6 cm. Diam., .25 mm. Moment of inertia, 133 g. $\overline{cm}^2$.

No. of Swing.	Exp. *A*. Initial Amp. 35°.0. Temp. 20° 5.		Exp. *B*. Initial Amp. 35°.4. Temp. 20°.5.		Exp. *C*. Initial Amp. 38°.0. Temp. 20°.8.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	5.236	30.4	5.244	30.8	5.247	33.8
10	5.234	26.3	5.239	26.5	5.241	29.3
15	5.230	22.9	5.234	23.2	5.235	25.4
20	5.227	20.0	5.231	20.1	5.232	22.1
25	5.224	17.5	5.228	17.6	5.230	19.3
30	5.221	15.5	5.226	15.3	5.228	16.9
35	5.219	13.6	5.225	13.5	5.225	14.9
40	5.217	11.9	5.222	11.9	5.222	13.1
45	5.216	10.6	5.220	10.5	5.220	11.5
50	5.214	9.3	5.218	9.2	5.217	10.2
55	5.213	8.3	5.217	8.2	5.216	9.0
60	5.213	7.3	5.216	7.3	5.215	7.9
65	5.213	6.5	5.215	6.5	5.214	7.0
70	5.212	5.9	5.214	5.7	5.214	6.3
75	5.211	5.3	5.213	5.1	5.214	5.6
80	5.210	4.7	5.213	4.6	5.214	5.0
85	5.209	4.1	5.213	4.0	5.213	4.4
90	5.209	3.7	5.213	3.6		
95	5.208	3.3	5.212	3.2		

The results of these experiments are shown graphically in Fig. 2, curves *A* and *B*. In this figure, the amplitudes are plotted as abscissæ and the periods as ordinates. In curve *A*, we have the results of experiment *A* plotted and in curve *C*, the results of *B* and *C*. It will be noticed that in all three cases the points lie very nearly in a straight line. The straight line marked *A* lies somewhat below that marked *B* but the slope of the two lines is about the same. This would indicate that the continued vibration which the wire underwent, before the observations in *B* were taken, has increased the period for all the amplitudes from the maximum to the lowest for which readings were taken. Curve *B* shows that the points for experiments *B* and *C* lie very nearly on the same straight line.

To determine whether or not the results are the same when the wire is kept in continual vibration at a larger amplitude, the experiment was repeated using an initial amplitude of about 190°. This amplitude is slightly under that permitted by the elastic limit of the wire.

The results of these experiments are shown in Fig. 3. The results are quite similar to those found in the preceding experiment. Fig. 3 shows the curve for experiments *B* and *C* lying above that for *A* and the curves for *B* and *C* coinciding. These experiments indicate that the

wire for the time being at least has been put in a definite state. It was determined, as a result of these experiments, that all subsequent series of observations should be taken after the wire had been kept in continuous vibration for a period of from twenty minutes to half an hour.

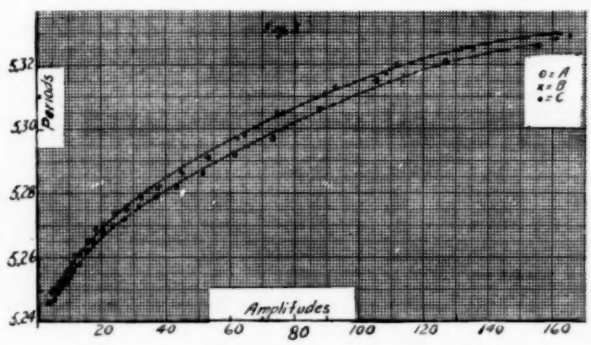

Fig. 3.

TABLE IV.

Length of wire, 106 cm. Diam., .25 mm. Moment of inertia, 133 g. cm^2.

No. of Swing.	Exp. *A*. Initial Amp. 184°.8. Temp. 23°.2.		Exp. *B*. Initial Amp. 190°.5. Temp. 23°.6.		Exp. *C*. Initial Amp. 195°.0. Temp. 23°.7.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	5.326	156.5	5.328	161.5	5.329	165.8
10	5.321	127.9	5.325	131.7	5.235	135.8
15	5.315	105.5	5.318	108.2	5.320	111.8
20	5.306	87.6	5.311	89.7	5.313	92.2
25	5.297	73.1	5.305	74.7	5.305	76.8
30	5.292	61.3	5.297	62.2	5.298	64.4
35	5.286	51.6	5.291	52.5	5.291	53.9
40	5.282	43.7	5.287	44.2	5.286	45.7
45	5.279	37.2	5.281	37.6	5.282	38.6
50	5.276	31.8	5.279	32.0	5.278	32.9
55	5.272	27.1	5.276	27.3	5.275	28.1
60	5.270	23.3	5.274	23.6	5.274	24.2
65	5.268	20.3	5.271	20.5	5.270	20.8
70	5.265	17.4	5.269	17.6	5.269	17.9
75	5.262	15.2	5.265	15.3	5.265	15.5
80	5.258	13.2	5.261	13.3	5.262	13.5
85	5.256	11.6	5.258	11.6	5.261	11.8
90	5.254	10.3	5.256	10.1	5.258	10.4
95	5.252	9.0	5.254	8.8	5.256	9.0
100	5.251	7.9	5.253	7.8	5.253	7.9
105	5.249	6.8	5.252	6.8	5.251	7.0
110	5.247	6.0	5.251	6.3	5.250	6.1
115	5.247	5.2	5.250	5.6	5.250	5.3
120	5.246	4.6	5.250	4.9	5.249	4.7
125	5.246	4.1	5.249	4.3		
130	5.246	3.6				

The logarithmic decrements for the various amplitudes were worked out for the above experiment and these logarithmic decrements plotted against the corresponding amplitudes in order to determine the effect of continued vibration on the logarithmic decrement. The results are given in Table V.

TABLE V.

Exp. *A*.			Exp. *B*.			Exp. *C*.		
No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.
8	.0178	156.5	8	.0179	161.5	8	.0174	165.8
10	.0170	127.9	10	.0174	131.5	10	.0171	135.8
10	.0166	105.5	10	.0167	108.3	10	.0168	111.8
10	.0160	87.6	10	.0161	89.7	10	.0163	92.2
10	.0155	73.1	10	.0159	74.7	10	.0156	76.8
10	.0151	61.3	10	.0153	62.2	10	.0154	64.4
10	.0147	51.6	10	.0148	52.5	10	.0149	53.9
10	.0142	43.7	10	.0144	44.2	10	.0145	45.7
10	.0138	37.3	10	.0140	37.6	10	.0143	38.6
20	.0137	31.8	20	.0136	32.0	20	.0139	32.9
20	.0132	27.1	20	.0132	26.3	20	.0135	28.1
20	.0128	23.3	20	.0130	23.6	20	.0132	24.2
20	.0126	20.3	20	.0127	20.5	20	.0129	20.9
20	.0123	17.6	20	.0125	17.6	20	.0127	17.9
20	.0122	15.3	20	.0124	15.3	20	.0123	15.5
20	.0116	13.2	20	.0121	13.3	20	.0119	13.5
20	.0114	11.6	20	.0120	11.6	20	.0118	11.8
20	.0111	10.3	20	.0116	10.1	40	.0117	10.4
20	.0115	7.9	40	.0108	7.8	40	.0115	7.9
20	.0113	6.0	40	.0106	5.6	40	.0112	6.1
20	.0113	4.6				40	.0109	4.7
20	.0113	3.6						

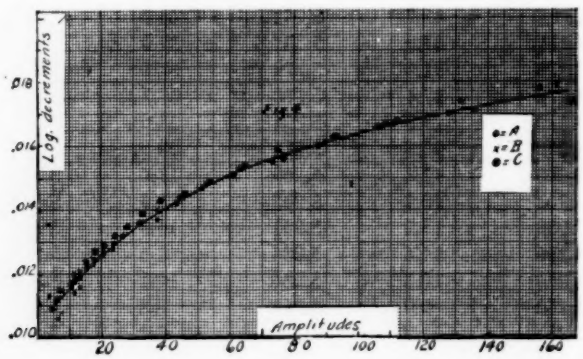

Fig. 4.

In the above table, the first, fourth and seventh columns give the number of swings used in determining the mean logarithmic decrement. The results have been plotted and are shown in Fig. 4. It is seen that

all three sets of points lie very nearly on the same curve, indicating that the continued vibration of the wire has had little if any effect on the logarithmic decrement.

IV. Increasing Amplitudes.

Having settled upon the method of procedure, the first experiment undertaken was to determine the effect of the initial amplitude upon the behavior of the wire. For this purpose, a wire of .25 mm. diameter that had not been used in any experiment before was hung in the bracket with a 50 g. suspension and allowed to hang in position for one day before taking the first series of observations. In the succeeding four days, thereafter, four series of observations were taken, one on each day. On the first, an initial amplitude of 34°.4 was used. On the second the initial amplitude was 84°.2, on the third 131°.0 and on the last 189°.5. In each case the wire was kept vibrating at the initial amplitude for a period of from 20 to 30 minutes before beginning the experiment. The data are contained in Table VI.

Table VI.

Length of wire, 103.6 cm. Diam., .25 mm. Moment of inertia, 133 g. cm².

No. of Swing.	Exp. *A*. Init. Amp. 34°.4. Temp. 20°.5.		Exp. *B*. Init. Amp. 82.°4. Temp. 22°.0.		Exp. *C*. Init. Amp. 131°.0. Temp. 22°.5.		Exp. *D*. Init. Amp. 190°.5. Temp. 23°.6.	
	Per.	Amp.	Per.	Amp.	Per.	Amp.	Per.	Amp.
5	5.244	30.8	5.279	71.9	5.305	112.5	5.328	161.5
10	5.239	26.5	5.272	60.6	5.299	93.3	5.325	131.7
15	5.234	23.2	5.266	51.3	5.291	77.8	5.318	108.2
20	5.231	20.1	5.261	43.7	5.284	65.1	5.311	89.7
25	5.228	17.6	5.256	37.4	5.279	54.8	5.305	74.7
30	5.226	15.3	5.253	32.1	5.273	46.5	5.297	62.2
35	5.225	13.5	5.250	27.6	5.271	39.4	5.291	52.5
40	5.222	13.5	5.247	23.9	5.269	33.7	5.287	44.2
45	5.220	10.5	5.245	20.7	5.263	28.8	5.281	37.6
50	5.218	9.2	5.243	18.0	5.261	24.8	5.279	32.0
55	5.217	8.2	5.241	15.8	5.258	21.3	5.276	27.3
60	5.216	7.3	5.240	13.9	5.254	18.5	5.274	23.6
65	5.215	6.5	5.238	12.0	5.253	16.1	5.271	20.5
70	5.214	5.7	5.236	10.6	5.250	13.9	5.269	17.6
75	5.213	5,1	5.233	9.4	5.248	12.2	5.265	15.3
80	5.213	4.6	5.231	8.3	5.247	10.6	5.261	13.3
85	5.213	3.6	5.229	7.3	5.245	9.4	5.258	11.6
90	5.213	3.6	5.227	6.5	5.244	8.2	5.256	10.1
95	5.212	3.2	5.227	5.7	5.242	7.2	5.254	8.8
100			5.226	5.1	5.240	6.4	5.253	7.8
105			5.225	4.5	5.238	5.7	5.252	6.8
110			5.225	4.0	5.237	5.0	5.251	6.2
115			5.224	3.6			5.250	5.6
120							5.250	4.9
125							5.249	4.3

In Fig. 5 the curves *A*, *B*, *C* and *D* are the period-amplitude curves for experiments *A*, *B*, *C* and *D*. It will be observed that the periods in each successive curve are higher than the corresponding periods of the next preceding curve. These curves are in many respects different from those obtained by Guthe and Sieg with the platinum-iridium wires. In the corresponding experiment with the platinum-iridium wires, while

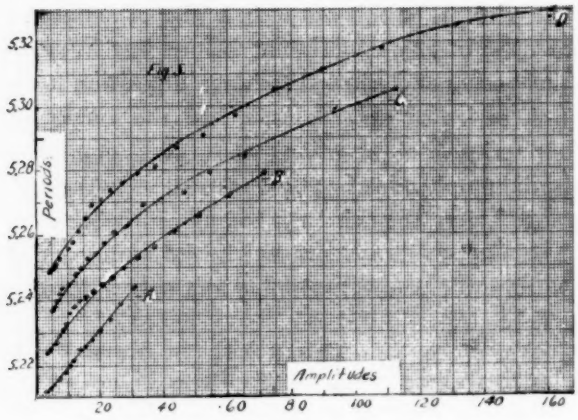

Fig. 5.

the curves for the experiments performed with the large initial amplitudes lay above those for the small amplitudes, the curves seemed to start from a common point, which is not the case with the curves for the bismuth wires. It was also noticed in the experiments with the platinum-iridium wires that at large amplitudes the periods approached a certain maximum value. In this experiment and in all others with the bismuth wires, it was found that, while the curves became less steep with larger amplitudes, no maximum was reached. However, in the case of the bismuth wires, the observations are limited by the low elastic limit to amplitudes that are very low compared to those used with the platinum-iridium wires, so that it may be that the curves shown in the figure correspond to those portions of the curves for the platinum-iridium wires, where the rate of increase in the period is greatest. Also it must be remembered that in the experiment with the platinum-iridium wires the wires were not kept in continuous vibration before taking the observations.

It will be noticed that the curves for the bismuth wires have very much the same shape and slope. The chief difference in the successive curves seems to be one of displacement, the direction of displacement being upward for the larger amplitudes.

The Logarithmic Decrement.—The logarithmic decrements for these

observations were computed and plotted as ordinates with the corresponding amplitudes as abscissæ. The results are given in Table VII. and the curves shown in Fig. 6. The curves *A*, *B*, *C* and *D* correspond to experiments *A*, *B*, *C* and *D*.

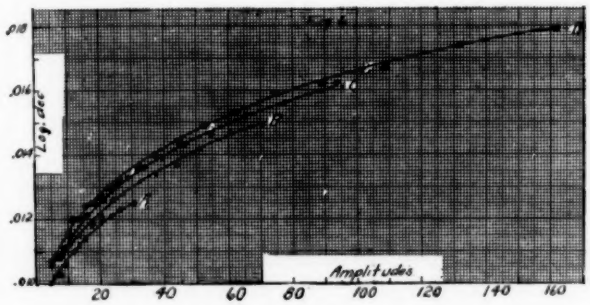

Fig. 6.

It is found that in all the curves there is a marked decrease in the logarithmic decrement with decrease in amplitude. Here, as in the period amplitude curves, we find that each successive curve lies above the next preceding one. These results would indicate that the greater twist of the wire through the large amplitudes increases the internal friction and that this increase in internal friction is not confined to the swings of larger amplitude but persists through the smaller amplitudes.

In the case of the logarithmic decrement-amplitude curves we again notice a striking difference from the corresponding curves for the platinum-iridium wires. In those wires, the logarithmic decrement at first, starting with the large amplitudes, increased slowly to a maximum value and then decreased very rapidly. It may be that here, too, the curve for the bismuth wire corresponds to that portion of the logarithmic decrement curve for the platinum-iridium wires where the amplitudes are small and the values of the decrement are decreasing.

Period and Vibration Number Curve.—For the last of the above four observations, that in which the initial amplitude was greatest, the periods were plotted as ordinates and the corresponding vibration numbers as abscissæ. For lack of space this curve is not shown here. It was found however that this curve also differs from the corresponding curve for the platinum-iridium wire.[1] In the latter case, the curve starts out almost parallel to the x-axis, then bends rapidly downward, and finally again approaches a horizontal direction. Neither in this, nor in any other curve of the same sort for the bismuth wire, could the tendency to start out in a direction parallel to the x-axis be detected. In other words, the curve is convex downward throughout its length. However, here again, it is

[1] Sieg, loc. cit., p. 425.

TABLE VII.

Exp. *A*.			Exp. *B*.		
No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.
8	.0125	30.8	8	.0148	70.9
10	.0123	26.5	10	.0147	60.6
20	.0121	23.2	10	.0142	51.3
20	.0119	20.1	10	.0137	43.7
20	.0118	17.6	10	.0134	37.4
20	.0114	15.3	10	.0132	32.1
20	.0112	13.5	20	.0128	27.6
40	.0110	11.9	20	.0126	23.9
40	.0107	9.2	20	.0121	20.7
40	.0103	7.3	20	.0118	18.0
40	.0102	5.7	20	.0115	13.9
40	.0100	4.6	20	.0112	10.6
			40	.0109	8.3
			40	.0106	5.1
Exp. *C*.			Exp. *D*.		
No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.
8	.0164	112.5	8	.0179	161.5
10	.0162	93.3	10	.0174	131.7
10	.0156	77.8	10	.0167	108.2
10	.0152	65.1	10	.0161	89.7
10	.0146	54.8	10	.0159	74.7
10	.0143	46.5	10	.0153	62.2
10	.0139	39.4	10	.0148	52.5
10	.0136	33.7	10	.0144	44.2
10	.0133	28.8	10	.0140	37.6
20	.0130	24.8	20	.0136	32.0
20	.0126	21.3	20	.0132	26.3
20	.0125	18.5	20	.0130	23.6
20	.0121	16.1	20	.0127	20.5
40	.0120	13.9	20	.0125	17.6
40	.0115	10.6	20	.0124	15.3
40	.0111	8.2	20	.0121	13.3
40	.0109	6.4	20	.0120	11.6
			20	.0116	10.1
			40	.0108	7.8
			40	.0106	5.6

that portion of the curve for the platinum-iridium wires that corresponds to the higher amplitudes that is missing in the curves for the bismuth wires.

In one other respect does the action of the bismuth wire differ from the platinum-iridium wire. In the case of the platinum-iridium wires, it was found that the wire made a larger number of vibrations in coming to rest when started with small amplitudes than when started with

larger amplitudes. Thus when the platinum-iridium wire was started with an initial amplitude of 364.3 degrees, it required 125 swings to fall to 5 degrees, while, when the wire was started with an initial amplitude of 14.3 degrees, 200 swings were required for the amplitude to fall to 6.15 degrees.[1] This same effect can be noted in the bismuth wires but not in anything like the same degree. In experiment *A* above, in which the initial amplitude was 34°.4, it required 89 vibrations for the amplitude to fall to 3.6 degrees. In experiment *D* in which the initial amplitude was 190.5 degrees, it required only 80 vibrations to cover the same interval.

V. Effect of Changing the Moment of Inertia.

The next experiment undertaken was to determine the effect of changing the moment of inertia of the vibrating system on the period-amplitude curve and on the logarithmic decrement curve. For this purpose, a wire .5 mm. in diameter and 95.7 cm. in length was used. Three experiments

Table VIII.

Diam. of wire, .5 mm. Length, 95.7 cm.

No. of Swing.	Exp. *A*. Mom. of Iner. 1,299 g. cm². Mass 150 g. Initial Amp. 89°.4. Temp. 26°.0.		Exp. *B*. Mom. of Iner. 1,882 g. cm². Mass 200 g. Initial Amp. 86°.5. Temp. 26°.0.		Exp. *C*. Mom. of Iner. 1,456 g. cm². Mass 200 g. Initial Amp. 82°.9. Temp. 25°.8.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	4.214	77.1	5.075	74.2	4.471	71.3
10	4.207	64.2	5.068	61.8	4.464	59.4
15	4.200	54.0	5.062	51.6	4.459	49.6
20	4.193	45.3	5.054	43.3	4.453	41.6
25	4.186	38.2	5.048	36.5	4.449	35.1
30	4.181	33.3	5.043	30.8	4.445	29.8
35	4.177	27.5	5.037	26.1	4.441	25.2
40	4.174	23.5	5.033	22.1	4.437	21.7
45	4.171	19.9	5.029	19.0	4.433	18.4
50	4.168	17.2	5.026	16.3	4.428	15.7
55	4.165	14.8	5.024	14.0	4.425	13.5
60	4.162	12.7	5.022	12.1	4.423	11.8
65	4.160	10.9	5.019	10.5	4.419	10.2
70	4.158	9.5	5.015	9.1	4.417	8.9
75	4.156	8.1	5.012	8.0	4.415	7.8
80	4.155	7.1	5.010	6.9	4.413	6.8
85	4.153	6.2	5.006	6.1	4.410	5.9
90	4.151	5.5	5.004	5.3	4.409	5.2
95	4.150	4.8	5.002	4.7	4.407	4.2
100	4.148	4.2	5.000	4.1	4.405	3.9
105	4.147	3.7	5.000	3.7		
110	4.147	3.2				

[1] Guthe and Sieg, loc. cit., p. 617.

were carried out, A, with a suspended weight having a mass of 150 g. and a moment of inertia of 1,299 g. $\overline{cm}^2$; B, with a weight of 200 g. and a moment of inertia of 1,882 g. $\overline{cm}^2$; and C, with a weight of 200 g. and a moment of inertia of 1,456 g. $\overline{cm}^2$. By comparing experiments A and B, it was possible to determine the effect of a change in both the moment of inertia and the mass of the suspended weight. By comparing B and C, it was possible to determine the effect of changing the moment of inertia only. The results are shown in Table VIII.

The period-amplitude curves are of the same general appearance as those found in the experiments described above. In these experiments, it was found necessary to use smaller amplitudes since the wire had a lower elastic limit than the one of smaller cross-sectional area used in the earlier experiments.

TABLE IX.

B factor = .8813. A factor = 1.0615.

Old Period.	New Period.	Amplitude.	Old Period.	New Period.	Amplitude.
5.975	4.473	74.2	4.214	4.473	77.1
5.068	4.466	61.8	4.207	4.465	64.2
5.062	4.461	51.6	4.200	4.458	54.0
5.054	4.454	43.3	4.193	4.451	45.3
5.048	4.449	36.5	4.186	4.443	38.1
5.043	4.444	30.8	4.181	4.438	33.4
5.037	4.439	26.1	4.177	4.434	27.5
5.033	4.436	22.1	4.174	4.430	23.5
5.029	4.432	19.0	4.171	4.427	19.9
5.026	4.429	16.3	4.168	4.424	17.1
5.024	4.428	14.0	4.165	4.421	14.8
5.022	4.426	12.2	4.162	4.418	12.7
5.019	4.423	10.3	4.160	4.416	10.9
5.015	4.420	9.1	4.158	4.413	9.5
5.012	4.417	8.0	4.156	4.411	8.1
5.010	4.415	6.9	4.155	4.410	7.1
5.006	4.412	6.1	4.153	4.408	6.2
5.004	4.410	5.3	4.151	4.406	5.5
5.002	4.408	4.7	4.150	4.405	4.8
5.000	4.407	4.2	4.148	4.404	4.2
5.000	4.407	3.7	4.147	4.402	3.7
			4.147	4.402	3.2

In order to afford a means of comparison for the three curves, the three series of observations were all reduced to the same period, that of experiment C. To do this, the three curves were produced until they cut the y-axis and from these points of intersection, a reduction factor was obtained. All the values for the periods of the curve to be reduced were then multiplied by this factor. This operation will have the effect of making the two curves to be compared start with the same period at

zero amplitude. In Table IX. are shown the results for experiments A and B reduced in the manner described above so as to correspond to that of C.

These reduced values were plotted and the curves are shown in Fig. 7. Curve A represents the results for experiment A and curve C the results for experiments B and C. It is found from Fig. 7, that the curve for

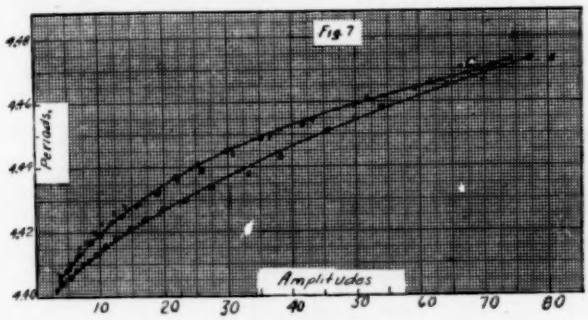

Fig. 7.

experiment A starts and finishes at about the same places as that for experiment C, but the middle portion lies quite a little below that for C. In other words, the curve for A is much flatter than that for C. On the other hand, the curves for B and C very nearly coincide. In fact, the variation in position of the two sets of points in the combined curve is no greater than the variation in the position of the points in either curve considered alone. When we remember that in observations A and C, both the mass of the vibrating system and its moment of inertia were varied, while in B and C only the moment of inertia was varied, we must conclude that, if we keep the stress to which the wire is subjected constant, the period will be the same function of the amplitude no matter what the moment of inertia, while if the stress as well as the moment of inertia is varied, the period will be a different function of the amplitude.

From the formula for the period, $T = 2\pi\sqrt{I/T}$, we should expect the reduction factor in each case to be equal to the square root of the ratio of the corresponding moments of inertia. The reduction factor for reducing observation A to observation C is 1.061. The value of $\sqrt{I_A/I_B}$ is 1.059. The reduction factor for reducing C to B is .881 while the value of $\sqrt{I_B/I_C} = .879$. When we remember that the factors here given were obtained from the curves, the agreement is remarkably close.

For lack of space the curves showing the relation between the logarithmic decrements and amplitudes in this experiment are not shown but it was found that here again we have a very good agreement between the curves for experiments B and C while the curve for experiment A differs to quite a marked extent from the other two, the curve for experiment A

lying below that for experiments *B* and *C*. This difference is especially noticeable in that portion of the curve corresponding to the larger amplitudes. This indicates that the shape of the logarithmic decrement-amplitude curve is also independent of the moment of inertia of the suspended weight but does depend upon the stress to which the wire is subjected. The effect of increasing the mass of the vibrating system is to increase the values for the logarithmic decrement especially for the higher amplitudes.

VI. The Effect of Variation in the Length of the Wire.

For determining the effect of a variation in the length of the wire upon the period-amplitude curve, a wire .25 mm. in diameter was used. Observations were taken using three different lengths: the first 104.8 cm., the second 82.0 cm. and the third 62.6 cm.

In all three cases, a cylinder weighing 50 g. and having a moment of inertia of 133 g. $\overline{cm}^2$. was used. The results obtained are given in Table X.

Table X.

Diam., .25 mm. Moment of inertia, 133 g. $\overline{cm}^2$.

No. of Swing.	Exp. *A*. Length 104.8 cm. Initial Amp. 175°.4. Temp. 26°.0.		Exp. *B*. Length 84.0 cm. Initial Amp. 136°.9. Temp. 26°.5.		Exp. *C*. Length 62.6 cm. Initial Amp. 105°.3. Temp. 27°.0.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	5.306	149.1	4.734	116.1	4.131	88.8
10	5.296	121.5	4.725	94.5	4.126	72.3
15	5.290	100.0	4.717	77.2	4.120	59.6
20	5.283	82.6	4.710	63.7	4.115	49.2
25	5.277	68.7	4.704	52.7	4.109	40.8
30	5.271	57.2	4.699	43.9	4.104	34.1
35	5.265	47.9	4.695	36.6	4.100	28.5
40	5.260	40.3	4.690	30.7	4.097	24.0
45	5.256	33.9	4.686	25.8	4.094	20.2
50	5.252	28.9	4.682	21.9	4.091	17.3
55	5.248	24.5	4.679	18.5	4.089	14.7
60	5.245	21.0	4.675	15.8	4.085	12.6
65	5.242	17.8	4.673	13.6	4.082	10.8
70	5.239	15.4	4.670	11.6	4.080	9.2
75	5.237	13.3	4.668	9.9	4.078	8.1
80	5.235	11.5	4.667	8.6	4.076	7.0
85	5.233	9.9	4.666	7.4	4.075	6.1
90	5.232	8.6	4.664	6.5	4.073	5.3
95	5.231	7.5	4.663	5.6	4.072	4.5
100	5.230	6.5	4.662	4.9	4.070	4.0
105	5.230	5.6	4.661	4.3	4.069	3.5
110	5.229	5.0			4.068	3.1
115	5.228	4.4			4.067	2.7
120	5.227	3.8				

In this case, as in the determination of the effect of a change in moment of inertia, it is necessary to reduce all experiments to the same period in order to obtain a basis for comparison. Experiment B was chosen as the basis for the comparison and the periods in the other two series reduced to this. The factors for reduction were obtained in the same way as before, that is, by determining the points of intersection of the curves with the y-axis. In this way, the curves will all start with the same period at zero amplitude. The reduced values for experiments A and C are given in Table XI.

TABLE XI.

A factor = .8912. C factor = 1.1458.

Old Period.	New Period.	Amplitude.	Old Period.	New Period.	Amplitude.
5.306	4.729	149.1	4.131	4.733	88.8
5.296	4.720	121.5	4.126	4.727	72.5
5.290	4.715	100.0	4.120	4.720	59.6
5.283	4.708	82.6	4.115	4.715	49.2
5.277	4.703	68.7	4.109	4.708	40.8
5.271	4.698	57.2	4.104	4.702	34.1
5.265	4.692	47.9	4.100	4.698	28.5
5.260	4.688	40.3	4.097	4.694	24.0
5.256	4.684	33.9	4.094	4.691	20.2
5.252	4.680	28.9	4.091	4.687	17.3
5.248	4.677	24.5	4.089	4.684	14.7
5.245	4.674	21.0	4.085	4.680	12.6
5.242	4.672	17.8	4.082	4.677	10.8
5.239	4.669	15.4	4.080	4.675	9.2
5.237	4.667	13.3	4.078	4.672	8.1
5.235	4.665	11.5	4.076	4.670	7.0
5.233	4.664	9.9	4.075	4.669	6.1
5.232	4.663	8.6	4.073	4.667	5.3
5.231	4.662	7.5	4.072	4.665	4.5
5.230	4.661	6.5	4.070	4.663	4.0
5.230	4.661	5.6	4.069	4.662	3.5
5.229	4.660	5.0	4.068	4.661	3.1
5.228	4.659	4.4	4.067	4.660	2.7
5.227	4.658	3.8			

In reducing the results of experiments A and C to compare with those of B, we should expect the factor in each case to be equal to the ratio of the square roots of the lengths, since the period of vibration varies as the square root of the length. The factor for the reduction of the data of experiment A to those of experiment B is. 8912. The value $\sqrt{L_B/L_A}$ is .8846, giving a difference of .0066. The factor for the reduction of C to B is 1.1458. The value of $\sqrt{L_B/L_A} = 1.1445$, giving a difference of .0013.

For lack of space the curves for these results are not shown but it was found that the curves did not coincide in this case as they did in the moment of inertia experiment. The curves for the shorter lengths were found to be much steeper than for the longer length. The amounts of decrease in period were about the same in all three cases. It must be remembered, however, that in this experiment we are not comparing corresponding points in the wire. That is to say, the point whose arc of vibration we are measuring in experiment *C* would vibrate through a much larger arc for the same amount of twist in the wire, if the point were at the end of a longer wire such as we have in experiment *A*. For this reason, a reduction factor must be applied to the amplitudes as well as to the period. These reduction factors were obtained by comparing the amplitudes corresponding to the same period in the two curves under consideration. All the amplitudes in the given experiment were then multiplied by this reduction factor. In this reduction, experiment *B* was taken as the standard and the amplitudes of *A* and *C* reduced so as to correspond. The reduced results for experiments *A* and *C* are shown in Table XII.

TABLE XII.

Experiment *A*. Factor = .732.			Experiment *C*. Factor = —.		
Old Amp.	New Amp.	Period.	Old. Amp.	New Amp.	Period.
149.1	110.1	4.729	88.8	128.7	4.733
121.5	88.9	4.720	72.5	105.1	4.727
100.0	73.2	4.715	59.6	86.4	4.720
82.6	60.5	4.708	49.2	71.1	4.715
68.7	50.3	4.703	40.8	59.2	4.708
57.2	42.4	4.698	34.1	49.4	4.702
47.9	35.1	4.692	28.5	41.3	4.698
40.3	29.5	4.688	24.0	34.8	4.694
33.9	24.8	4.684	20.2	29.3	4.691
28.9	21.1	4.680	17.3	25.0	4.687
24.5	17.9	4.677	14.7	21.3	4.684
21.0	15.4	4.674	12.6	18.3	4.680
17.8	13.0	4.672	10.8	15.7	4.677
15.4	11.3	4.669	9.2	13.3	4.675
13.3	9.6	4.667	8.1	11.7	4.672
11.5	8.1	4.665	7.0	10.2	4.670
9.9	7.2	4.664	6.1	8.8	4.669
8.6	6.3	4.663	5.3	7.7	4.667
7.5	5.5	4.662	4.5	6.5	4.665
6.5	4.7	4.661	4.0	5.8	4.663
5.6	4.1	4.661	3.5	5.1	4.662
5.0	3.7	4.660	3.1	4.5	4.661
4.4	3.2	4.659	2.7	3.9	4.660
3.8	2.8	4.658			

The curves obtained from these results together with the original curve for experiment *B* are shown in Fig. 8. We see from that figure that we have a remarkably close agreement between the three sets of points.

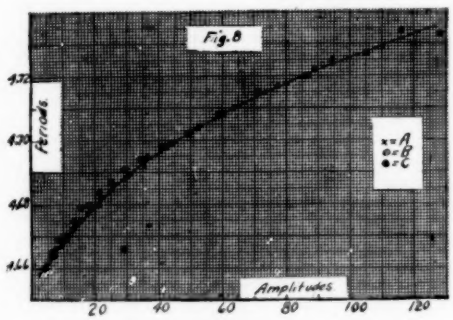

Fig. 8.

If the curves for these experiments were straight lines, we would expect the reduction factors for the amplitudes to be equal to the ratio of the lengths. As it is they are quite different. The reduction factor for changing *A* to *B* is .732 while the ratio of the lengths is .782. The reduction factor for the amplitudes of *C* is 1.450, while the ratio of the lengths is 1.310.

We must conclude from the fact that by merely applying constant factors to the two variables in these observations, the three curves can be made to coincide, that the period in each case is the same function of the amplitude. It also indicates that the wire is uniform throughout its length and that its peculiar behavior is not due to any particular portion of the wire.

VII. The Effect of Variation in Diameter on the Period-Amplitude Curve and the Logarithmic Decrement-Amplitude Curve.

To determine the effect of varying the diameter of the wires upon the period-amplitude curve and the logarithmic decrement-amplitude curve, two wires of different diameter were used with suspended weights of such a mass that the stress to which the wire was subjected was the same in both cases.

In this experiment, the wires were of .25 mm. and .5 mm. in diameter respectively. Since the cross-sectional area of the first was one fourth that of the second, the mass of the suspended weight in the first case was one fourth that in the second. Masses of 50 g. and 200 g. respectively were used.

TABLE XIII.

No. of Swing.	Exp. A. Diam. .25 mm. Length 97.9 cm. Mass 50 g. Moment of Inertia 133 g. cm². Temp. 28°.1.		Exp. C. Diam. .5 mm. Length 97.9 cm. Mass 200 g. Moment of Inertia 1,456 g. cm². Temp. 26°.0.	
	Period.	Amplitude.	Period.	Amplitude.
5	5.320	142.4	4.521	72.9
10	5.309	114.7	4.514	60.7
15	5.299	92.7	4.509	50.7
20	5.292	75.6	4.503	42.8
25	5.285	62.1	4.499	35.9
30	5.279	51.2	4.496	30.5
35	5.274	42.4	4.491	25.8
40	5.268	35.3	4.487	22.2
45	5.263	29.6	4.483	18.8
50	5.259	24.9	4.478	16.1
55	5.256	21.2	4.475	13.8
60	5.253	17.9	4.472	12.1
65	5.251	15.4	4.468	10.4
70	5.249	13.1	4.466	9.1
75	5.246	11.3	4.464	8.0
80	5.243	9.3	4.462	7.0
85	5.241	8.4	4.459	6.0
90	5.239	7.2	4.458	5.3
95	5.238	6.2	4.457	4.6
100	5.237	5.4	4.456	4.0
105	5.236	4.8		
110	5.235	4.1		
115	5.235	3.6		
120	5.234	3.2		

In this experiment, it was necessary to use a smaller initial amplitude for the wire of large diameter than for the one of small diameter because of the lower elastic limit in the one case than in the other. Of course,

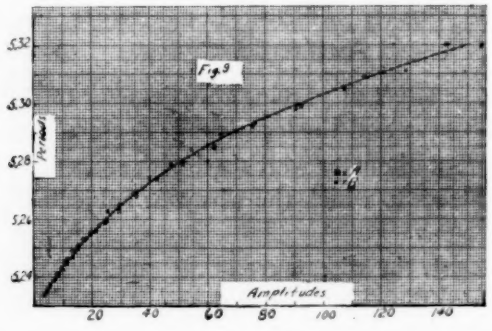

Fig. 9.

here again, we must apply a reduction factor to the periods in order to afford a basis of comparison. This was done, the factor being obtained

in the same way as before. Experiment *A* was taken as the standard and *B* reduced to correspond.

The factor for reducing the period should theoretically be equal to the inverse ratio of the squares of the diameters multiplied by the ratio of the square-roots of the moments of inertia. This operation gives a factor of 1.208, while that found from a comparison of the curves was 1.176.

If a second reduction factor be applied to the amplitudes it is found that the curves coincide. These reduced results are shown in Table XIV. and the two sets of points are plotted in Fig. 9.

TABLE XIV.

Factor = 2.12.

Old Amp.	New Amp.	Period.	Old Amp.	New Amp.	Period.
72.9	154.5	5.320	13.8	29.2	5.265
60.7	128.7	5.311	12.1	25.7	5.263
50.7	107.5	5.305	10.4	22.5	5.258
42.8	90.7	5.298	9.1	19.3	5.255
35.9	76.1	5.293	8.0	17.0	5.253
30.5	64.7	5.289	7.0	14.9	5.250
25.8	54.7	5.284	6.0	13.2	5.247
22.2	47.1	5.279	5.3	11.2	5.245
18.8	39.9	5.274	4.6	9.8	5.243
16.1	34.1	5.269	4.0	8.0	5.241

From the fact that the curves for the two observations can be made to coincide by applying constants to the two variables, we must conclude that when the stress to which the wire is subjected is kept constant, the period is the same general function of the amplitudes no matter what the diameter of the wire.

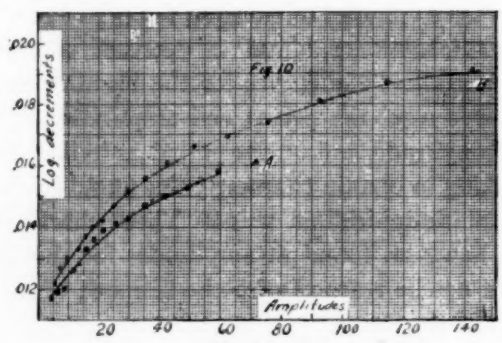

Fig. 10.

The Logarithmic Decrement-Amplitude Curve.—The logarithmic decrements in these two experiments were computed and plotted against the corresponding values of the amplitudes. The results of these computations are shown in Table XV. and the curves in Fig. 10.

TABLE XV.

Exp. A.			Exp. B.		
No. of Swings.	Log. Dec.	Amplitude.	No. of Swings.	Log. Dec.	Amplitude.
8	.0191	142.2	8	.0161	71.3
10	.0186	114.7	10	.0158	59.4
10	.0181	92.7	10	.0153	49.6
10	.0174	75.6	10	.0150	41.8
10	.0169	62.1	10	.0147	35.1
10	.0166	51.2	20	.0143	29.8
10	.0161	42.4	20	.0141	25.2
10	.0156	35.3	20	.0139	21.7
10	.0152	29.6	20	.0136	18.4
20	.0147	24.9	20	.0133	15.7
20	.0142	21.2	20	.0128	13.5
20	.0140	17.9	40	.0126	11.8
20	.0137	15.4	40	.0120	8.9
20	.0133	13.1	40	.0118	6.8
20	.0130	9.7			
20	.0127	7.2			
20	.0122	5.4			
20	.0117	4.1			

It will be noticed from Fig. 10 that while the logarithmic decrement curves start from about the same point, the one for the small diameter is steeper than that for the wire of larger diameter.

SUMMARY.

1. The elastic properties of bismuth wires have been investigated and it has been found that they display some of the same peculiar properties shown by the platinum-iridium wires investigated by Guthe and Sieg. When the wires were vibrated torsionally, the period of vibration and the logarithmic decrement were found to decrease enormously with the decrease in amplitude. The curves for the bismuth wires representing the relations between (*a*) the period and amplitude, (*b*) the logarithmic decrement and amplitude, and (*c*) the period and vibration number, were found to be lacking in certain characteristics found in the corresponding curves for the platinum-iridium wires. These characteristics were for the most part, however, found at amplitudes for the platinum-iridium wires that could not be reached with the bismuth wires, because of the low elastic limit of the latter.

2. It was found that, in two experiments in which the mass of the vibrating system was kept constant but the moment of inertia varied, the period-amplitude curves in the two cases could be made to coincide by applying a reduction factor to the periods in one set of observations.

This reduction factor was found to be equal to the ratio of the square roots of the moments of inertia used in the two experiments. This indicates that the mathematical relation between the period and amplitude does not depend upon the rapidity of motion of the vibrating system. The logarithmic decrement-amplitude curves in the two cases were found to coincide.

If the load as well as the moment of inertia was varied, it was found that when the proper reduction factor was applied to the periods, the two curves did not coincide, the curve corresponding to the vibrating system having the smaller mass being flatter than the other. The curves for the logarithmic decrement did not coincide, the greater values for the logarithmic decrement being found in the case in which the mass of the suspended weight was greatest.

3. When the length of the wire was varied, it was found that by applying reduction factors to the periods and amplitudes, the period-amplitude curves for the different experiments could be made to coincide, indicating that the period is the same general function of the amplitude in all cases. This also indicates that the wire is uniform throughout its length and that its peculiar behavior is not due to any particular section of the wire.

4. Two wires of different diameter were used with suspended weights of such a mass that the stresses to which the wires were subjected were the same in each case. In this experiment, also, by applying the proper factors to the amplitudes and periods the curves could be made to coincide. The logarithmic decrement curve for the smaller wire was found to lie above that for the wire of larger diameter, indicating a greater damping effect in the former.

In conclusion, the writer wishes to express his appreciation of the helpful advice and kindly interest of Prof. K. E. Guthe, at whose suggestion this investigation was undertaken.

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THE HALF-VALUE OF THE RADIOACTIVE DEPOSIT COLLECTED IN THE OPEN AIR.

BY FREDERIC A. HARVEY.

IT has been quite definitely settled by experiment that the radioactive deposit collected in the open air on a negatively charged wire consists of a mixture of the disintegration products of radium and thorium.[1] The proportion of the two active deposits varies, depending upon the number of hours of exposure, potential of the wire, the atmospheric conditions, as well as the locality. The question of the half-value however seems to be still an open one. S. J. Allen and Harvey find that the half-value is not a constant but varies between wide limits. Other observers have maintained that the half-value is constant or varies but slightly.

It is the object of the present paper to show, from theoretical considerations, that the half-value should vary; to show how it should vary, and to compare theory with experiment.

When a negatively charged body is exposed to thorium emanation the manner in which the active deposit collected on the wire decays depends on the number of hours of the exposure. If the exposure is short, *i. e.*, less than about 6 hours, the number of radioactive particles transforming per second increases during the first few hours (3 to 6 hours) then gradually decreases and finally decays regularly according to an exponential law. After this regular decay in activity begins, half-value is reached in about eleven hours. Even when the exposure is of 18 to 24 hours duration the activity does not begin at once to decay logarithmically but increases very slightly for about an hour, then begins gradually to decrease, and after three or four hours decays logarithmically with half-value 10.6 hours, until the activity disappears. The activity, at any time subsequent to the removal from the emanation of the wire on which the active deposit is collected, in terms of the initial activity, may be calculated from the equation

[1] H. A. Bumstead, Amer. Jour. Sc., July, 1904; H. M. Dadourian, Amer. Jour. Sc., Jan., 1905; S. J. Allen, Phil. Mag., Dec., 1904, and PHYS. REV., June, 1908; G. A. Blanc, Phil. Mag., Mar., 1907; D. Pacini, Phys. Zeit., Mar. 15, 1910; F. A. Harvey, PHYS. REV., Mar., 1909.

$$I = I_0 \frac{ae^{-\lambda_2 t} - be^{-\lambda_1 t}}{a - b},$$[1]

where t is the time, in seconds, from the instant of discharge of the wire, *i. e.*, from the instant when it ceases to collect active material from the transforming emanation, $\lambda_1 = .000208$, $\lambda_2 = .0000182$ and a and b are constants depending on the time of exposure T, to the emanation, in seconds.

$$a = \frac{1 - e^{-\lambda_2 T}}{\lambda_2}, \qquad b = \frac{1 - e^{-\lambda_1 T}}{\lambda_1}.$$

This theory has been tested experimentally by Miss Brooks,[2] Her results show good agreement between theory and experiment, except for exposures less than one hour. Miss Brooks obtained her results with exposures of 1, 2, 3, 4 and 6 hours. In comparing experimental with theoretical values she used the constants $\lambda_1 = .00021$ and $\lambda_2 = .0000175$. These values have since been superseded by the more accurate ones given above.

From the equation above, I have calculated the values which the activity on the wire would have at different times up to 50 hours after removal from the emanation. In order that the comparison with open air exposures may be made, calculations are for 3, 4, 12 and 22 hour exposures, these being frequent times used in open air experiments.

The results of this calculation are given in Table I. Figure 1 shows

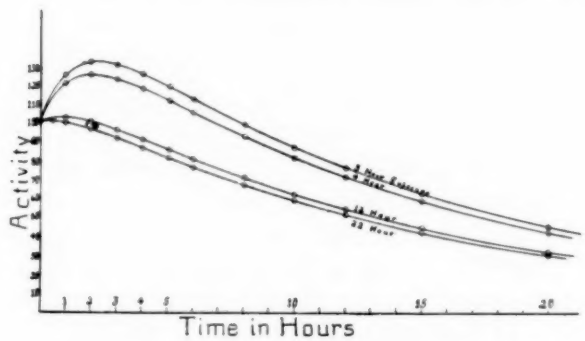

Fig. 1.

Curves showing rise and decay of the active deposit from thorium emanation for different lengths of exposure.

the same thing graphically. It is to be noted that the activity at first increases in intensity. Even for the 22 hour exposure the curve does

[1] Rutherford, Radioactivity, page 351 et sequa.

[2] Phil. Mag., Vol. 8, 1904, p. 373.

not become logarithmic until about 3½ to 4 hours. For this long exposure the activity increases for the first half hour.

TABLE I.

Time after Activation.	3 Hour Exposure.	4 Hour Exposure.	12 Hour Exposure.	22 Hour Exposure.
0 hours	100	100	100	100
½ hour				100.4
1 hour	124.8	120.5	102.4	99.8
2 hours	131.7	124.8	100.3	96.3
3 hours	130.3	122.7	96.0	91.6
4 hours	125.2	117.6	90.8	86.4
5 hours	118.8	111.4	85.6	81.3
6 hours	112.0	105.1	80.3	76.3
8 hours	98.8	92.6	70.9	67.0
10 hours	86.8	81.5	62.0	58.8
12 hours	76.3	71.5	54.4	51.6
15 hours	62.7	58.8	44.7	42.4
20 hours	45.3	42.3	32.2	30.5
30 hours	24.3	22.0	16.4	15.9
50 hours	6.3	5.9	4.5	4.3

The curve of decay for the active deposit from radium might also be calculated, but, owing to the fact that there is a larger number of transformation products, the calculation would be much more complicated. For this reason an experimental curve has been used. This experimental curve has been compounded with the curve shown in Fig. 1 for the four hour exposure and the result is shown in Table II. Table II. shows, then, for a four hour exposure, the decay of the activity on a wire where this activity consists of a mixture of radium and thorium disintegration products, in varying proportions. The activity is taken as 100 at the ten minute point, *i. e.*, ten minutes after the wire is removed from the emanation, as in open air experiments it takes about ten minutes before observations can be commenced. Also the radium A has practically all transformed by this time and the initial activity is thus much easier to determine. The abbreviations "Th" and "Ra" are used to indicate the disintegration products of thorium and radium respectively. Any column in Table II. will give the activity of a mixture of thorium and radium active deposits in the percentages indicated at the head of the column for the times given in the first column.

These values are plotted in Fig. 2 (circle points). From these curves for the decay of the mixture of active products it is a simple matter to determine the half-value period. The curve drawn in Fig. 2 with *X*'s shows the variation of the half-period. The variation is at first slight, *i. e.*, until about 30 per cent. is reached, then it begins to suddenly become

TABLE II.

Four Hour Exposure to a Mixture of Emanations.

Time.	Th, 100 Per Cent., Ra, 0 Per Cent.	Th, 0 Per Cent., Ra, 100 Per Cent.	Th, 10 Per Cent., Ra, 90 Per Cent.	Th, 20 Per Cent., Ra, 80 Per Cent.	Th, 30 Per Cent., Ra, 70 Per Cent.	Th, 40 Per Cent., Ra, 60 Per Cent.	Th, 50 Per Cent., Ra, 50 Per Cent.	Th, 60 Per Cent., Ra, 40 Per Cent.	Th, 70 Per Cent., Ra, 30 Per Cent.	Th, 80 Per Cent., Ra, 20 Per Cent.
0										
10 min.	100	100	100	100	100	100	100	100	100	100
20	104.3	87.4	98.1	90.8	92.5	94.3	96.0	97.7	99.4	101.1
40	110.8	68.5	72.8	77.0	81.1	85.4	89.6	93.9	98.1	102.4
60	115.8	48.9	55.6	62.3	69.0	75.6	82.3	98.1	95.8	102.5
80	118.2	33.5	41.8	50.4	58.9	67.4	75.9	84.3	92.8	101.3
100	119.6	22.3	32.1	41.7	51.5	61.2	70.9	80.7	90.4	100.2
120	119.8	14.6	25.1	35.7	46.2	56.7	67.2	77.7	88.2	98.7
3 hrs.	117.9	3.7	15.1	26.6	38.0	49.4	60.7	72.2	83.6	95.0
4	113.0	.6	11.8	23.1	34.3	45.6	56.5	68.0	79.4	90.5
5	107.0	.18	10.9	21.5	32.2	42.9	53.6	64.2	74.9	85.7
6	101.1	.16	10.3	20.3	30.4	40.5	50.6	60.7	70.8	80.9
8	89.1	0	8.9	17.8	26.7	35.6	44.5	53.4	62.3	71.3
10	78.3	0	7.8	15.7	23.5	31.3	39.1	47.0	54.8	62.7
12	68.7	0	6.9	13.7	20.6	27.5	34.3	41.2	48.0	54.8
15	56.6	0	5.7	11.3	17.0	22.6	28.3	33.9	39.6	45.2
18	46.2	0	4.6	9.2	13.8	18.5	23.1	27.7	32.3	36.9
20	40.7	0	4.1	8.1	12.2	16.3	20.4	24.4	28.5	32.5
30	21.1	0	2.1	4.2	6.3	8.4	10.5	12.7	14.8	16.9
50	5.8	0	.6	1.2	1.7	2.3	2.9	3.5	4.1	4.6

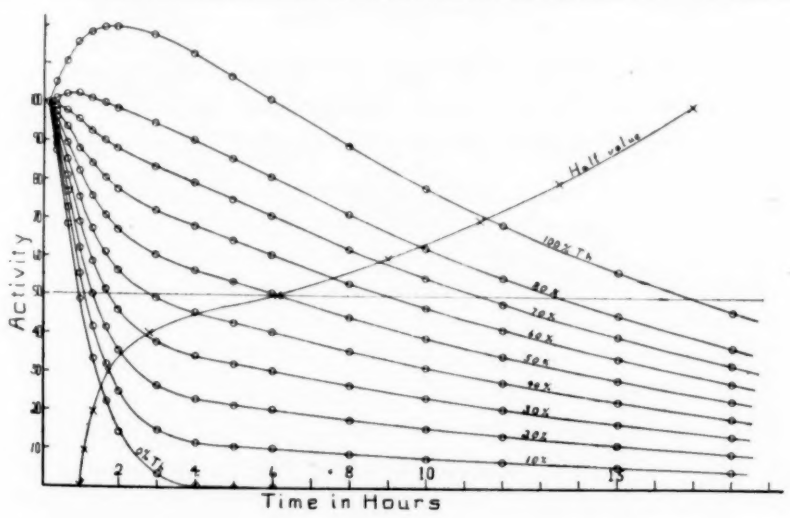

Fig. 2.

Theoretical curves for a mixture of radium and thorium active deposits, in varying proportions, for a four hour exposure. Half-value curve. The ordinates for the half-value curve are in per cent. with the same magnitude as the scale for the activity.

much greater and increases rapidly as the percentage of thorium active deposit increases until the limiting value, about 16½ hours is reached. From 20 per cent. to 40 per cent. the half-value increases from a little over 60 minutes to nearly three hours. Note that in Fig. 2 since the initial activity is taken as 100 at 10 minutes the time for the half-value is always ten minutes too large. The values corresponding to Table II. for the three hour and twelve hour exposures have been calculated and the corresponding curves plotted. From these curves the time to sink to half-value has been taken and these values are given in Table III. The same results are shown graphically in Fig. 3. The three hour and

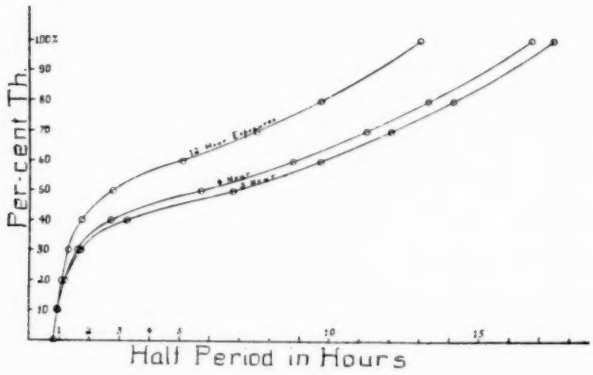

Fig. 3.

Curves showing variation of half-value of a mixture of active deposits with the percentage of thorium active deposit.

twelve hour exposures show the same general characteristics as the four hour exposure. In the three hour exposure there is a greater variation of the half-period and in the twelve hour it is slightly less. It is evident

TABLE III.

Per Cent. Th Active Deposit.	Time to Sink to Half-value.		
	3 Hr. Exposure.	4 Hr. Exposure.	12 Hr. Exposure.
0	47 min.	47 min.	47 min.
10	58	57	55
20	71	70	64
30	100	95	77
40	194	162	104
50	7 hr. 46 min.	5 hr. 42 min.	2 hr. 45 min.
60	9 hr. 40 min.	8 hr. 46 min.	5 hr. 5 min.
70	12 hr. 1 min.	11 hr. 12 min.	7 hr. 33 min.
80	14 hr. 5 min.	13 hr. 15 min.	9 hr. 41 min.
100	17 hr. 25 min.	16 hr. 41 min.	13 hr. 0 min.

that if the percentage of the active deposit from thorium varies up to 60 per cent. and 70 per cent., as experimental results have indicated it does, that the half-value must vary between very wide limits. In experiments performed by the writer active deposits containing a large percentage of thorium transformation products have been obtained and other observers, in particular Blanc,[1] have obtained even higher percentages.

TABLE IV.

No. Curve.	Hours Exposure.	Per Cent. Th.	Half-value.		Barometer.
			Observed.	Calculated.	
1	3	1 or 2	48 min.	45 min.	Rising rapidly.
2	3	1 or 2	48	43	Falling.
3	3	2	49	52	Rising rapidly.
4	3	3	51	53	Rising.
5	3	3.5	52	51.5	Steady.
6	3	5	53	45	Steady.
7	3	6	54	50	Falling slowly.
8	3	9.5	58	59	Steady.
9	3	10	58	53	Steady.
10	3	20	71	71	Steady.
11	4	1 or 2	48	47	Steady.
12	4	9	56	52	Nearly steady.
13	4	10	57	57	Steady.
14	4	19	69	50	Steady. Wind stopped.
15	12	8	54	46	Rising.
16	12	29	77	78	Steady.
17	13	10	54	52	Falling.
18	13	12	56	50	Falling.
19	13	62	5 hr. 45 min.	2 hr. 30 min.	Falling.
20	14	34	86	85	Steady.

It cannot be expected that there will be perfect agreement between theory and experiment in an exposure made in the open air. Calculations can only be made on a basis of a constant supply of emanations and in an open air exposure this is not likely to be realized. If the barometer changes rapidly just before or during an experiment the supply of thorium emanation will be changed, as has been shown by the writer.[2] If the wind changes in direction or speed it is probable that there will be a change in the ratio of the amounts of emanation in the air. Considering these facts, the agreement is surprisingly good. Table IV. shows the results of 20 experiments. Of these 20 experiments seven show only a slight difference from the calculated value; a difference of only

[1] Phil. Mag., 13, p. 378, March, 1907.

[2] Loc. cit.

one minute. This is within the error of calculation as the graphical method used will only give half values within about one minute. Thirteen results are within four minutes. When the agreement is not good it is nearly always plainly traceable to barometric change as may be seen from Table IV. There is only one experiment where the barometer is steady where the agreement is not at least fair and in that case there was a sudden decrease in the speed of the wind just before the start of the experiment.

The experiments by D. Pacini[1] do not show quite so good agreement with the above theory. Out of 22 experiments the barometer was steady or changed slightly in ten. None of the observations during which the barometer changed agree with the theory. This is to be expected as was pointed out above. Of the ten others, five show good agreement and one experiment where the barometer was steady was accompanied by a sudden change in the wind. There is no apparent reason why the other four results should not agree with the theory. Table V. shows a comparison between the half period found by Pacini and that calculated from the above theory for the nine experiments where agreement is to be expected. The percentages of thorium active deposit are of course the actual percentages on the wire and not the equilibrium values.

TABLE V.

No. Curve.	Hours Exposure.	Per Cent. Th.	Half-value.		Barometer.
			Observed.	Calculated.	
2	4	16.5	65	65	Slow rise.
11	4	15.5	61	64	Slow rise.
12	3½	12.1	59	59	Slow rise.
13	4	12.5	56	60	Slow rise.
14	4	8.5	52	55	Slow rise.
15	4	26.8	61	83	Slow rise.
18	4	12.6	53	60	Slight rise.
19	4	15.9	57	64	Slight rise.
20	4	11.7	56	58	Slight rise.

SUMMARY.

It is shown from theoretical considerations that the half-value of a mixture of the active deposits from radium and thorium emanations must vary between very wide limits (45 minutes to several hours) if the percentage of thorium active deposit varies from 0 per cent. to 50–70 per cent. It is further shown that when atmospheric conditions are

[1] Phys. Zeit., March 15, 1910, p. 209.

favorable, *i. e.*, steady barometer and only slight changes in the wind, very good agreement is obtained between theory and experiment for the mixture of active deposits collected on a negatively charged wire exposed in the open air.

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May 10, 1912.

THE MELTING OF ICE IN A MIXTURE OF ICE AND WATER.

BY E. E. SOMERMEIER.

THE results given in this paper were obtained by the author in connection with some calorimetric work in which a two-liter Dewar vacuum flask was used as a calorimeter. In testing out the possibilities of the flask as a calorimeter the determination of the latent heat of fusion of ice suggested itself and a series of determinations were run as follows:

1,600 grams of water were accurately weighed in the Dewar flask. A thermoneter graduated in .01 of a degree and readable by means of a telescope to .001 of a degree was inserted through a cork into the flask and temperature readings were taken from time to time, the contents of the flask being well mixed by shaking the flask between readings. After a series of readings had been obtained a weighed amount of ice was poured into the calorimeter and the latent heat of fusion determined by calculations based upon temperature changes and the amounts of water and ice used. Details of the calculations, data on the water equivalent of the calorimeter, thermometer corrections, etc., are omitted here as not being directly related to the subject matter of this article.

The method of work in so far as ice is concerned was as follows: Sticks of ice about one centimeter in diameter were frozen in glass tubes and after removal from the tubes the sticks were allowed to freeze dry in a room the temperature of which was several degrees below freezing. From 100 to 140 grams were then weighed in a cold Erlenmeyer flask. The flask was stoppered and immersed to the neck in a mixture of ice and water and allowed to remain for a considerable period of time. The mixture containing the flask and ice was then carried into the calorimeter room, the stopper of the flask removed, the neck grasped firmly by means of a handkerchief wrapped around it and the contents quickly poured into the calorimeter, the operation requiring only the fraction of a minute. The Erlenmeyer flask was then stoppered and set to one side while the temperature readings on the thermometer of the calorimeter flask were taken. The greater part of the ice quickly melted and within two minutes the contents of the calorimeter flask were within one to three degrees of the room temperature. With this temperature difference the temperature change per minute was only .0001 to .0003° and readings were taken

only at intervals of four or five minutes until a sufficient number had been taken to fix an end point and to establish a final rate, from which the temperature changes and radiation corrections were determined.

The Erlenmeyer flask in which the ice was weighed was wiped thoroughly dry on the outside and placed in the cold room and allowed to stand until at room temperature. It was then weighed and the difference between this weight and the weight with the ice sticks, taken as the weight of ice added to the calorimeter.

In the first series of determinations made in this way the amounts of ice used in a determination varied from 99 to 140 grams and the sticks were weighed at minus 7° and allowed to stand in the mixture of ice and water from 45 to 60 minutes. The idea of the writer was that the sticks, which were below the freezing temperature when put into the mixture, would in time attain practically the zero temperature of the mixture around the flask. The results obtained for the latent heat of fusion in this series of determinations are as follows: 79.88, 79.98 and 79.89 or an average of 79.92. These results were fairly satisfactory, but it seemed desirable to have an average of a larger number of determinations and a second series was begun.

In considering the results of the first determinations the only error that might be eliminated which suggested itself was the possibility that the sticks had not been allowed to stand in the mixture of ice and water sufficiently long to have arrived at approximately zero temperature. Accordingly in the second series, the sticks were immersed for a much longer period, the time varying from 70 to 200 minutes. The amounts of ice taken in this series of determinations varied as in the first series from about 100 grams up to about 140 grams. The second series of results checked fairly well among themselves, being as follows: 79.53, 79.56, 79.38, and 79.51, an average of 79.49. However, the average of this series does not agree at all with the average of the first series, being .4 lower. The greatest difference in conditions between the two series was that in the first the ice was weighed at minus 7° and immersed from 45 to 60 minutes, in the second it was weighed at minus 1½° to minus 3° and allowed to set from 70 to 200 minutes. Careful inspection of temperature rates, radiation corrections and points of manipulation in the two series showed no differences that would account for such a variation in results obtained and it seemed evident that the difference was in some way related to the difference in time of immersion in the mixture of ice and water. It was observed in the second series of results that the Erlenmeyer flasks after the ice was poured from them always contained a considerable amount of moisture, as much as a cubic centimeter

or so. Small amounts of moisture in the flasks might well have been due to breaking off of small particles from the cold brittle sticks when they were put into the flask and these small particles adhering to the sides of the flask upon melting would account for some moisture always being present, but the large amount present did not seem accountable for in this way. It did not seem possible that any of the ice had actually melted while immersed and that the disagreement in the calorimetric results was due to the fact that the sticks in the latter series when poured into the calorimeter were wet instead of dry. However in order to definitely determine this point tests were made as follows: Sticks of cold dry ice were suspended by fine wires in the Erlenmeyer flasks and the flasks immersed in ice and water as in the calorimeter experiments. Upon withdrawing the sticks drops of water were plainly visible upon the lower ends of the sticks and upon touching with filter paper appreciable amounts of water were absorbed by the paper.

Quantitative tests were then started in order to determine the amounts actually melted. In order to guard against errors due to any difference in water the same supply of distilled water recently boiled to expel air was used in freezing the sticks and in the freezing mixture. Details of treatment are as follows: The sticks of ice were handled by means of small wires looped around them and were allowed to remain in a cold porcelain evaporating dish or casserole until there was a certainty of their being frozen entirely dry if by any chance any melting should occur during the attachment of the wires. Stoppered weighing tubes were used for weighing the sticks, which as soon as weighed were transferred to long test tubes immersed in the mixtures of ice and water which were contained in glass cylinders. These test tubes were at the temperature of the room, which was below 0° C., so that both tubes and sticks when inserted into the mixture were colder than the mixture itself and would gradually warm up to the temperature of the surrounding mixture. During practically all of the tests the room temperature was considered below 0° C. and the mixture was continually freezing during the period of the tests. The sticks upon being removed from the tubes were touched lightly to cold filter paper to remove any drops of water hanging to them. They were then quickly placed in weighing tubes which were then closed and weighed and the loss observed and if the experiments were to be continued the sticks again transferred to the tubes in the mixture of ice and water.

Some of the results obtained are as follows:

TABLE I.

Test No.	Approx. Weight of Stick, Gms.	Room Temp. Beginning.	Room Temp. at End.	Time in Hours.	Volume in C.c. of Mixture of Ice and Water.	Ice Melted in Grams.	Ice Melted per Hour, Grams.
1*a*	87	− .8	−3.0	15½	1,500	2.17	.140
1*b*	85	−3.0	−1.0	24½		0.84	.034
1*c*	84	−1.0	+ .2	11		.57	.052
2*a*	35	− .8	−1.4	5	380	.46	.092
2*b*	35	−1.4	−3.0	11		.32	.031
2*c*	35	−1.4	−0.8	8½		.10	.012
2*d*	34	−0.6	−0.8	2¼	380	.03	.012
2*e*	34	−0.8	−1.0	12¾		.23	.018
2*f*	34	−1.0	−0.4	3		.15	.050
2*g*	34	−0.4	+0.2	7¾		.27	.035
3*a*	50	−2.0	+0.2	32½	1,550	1.00	.031
4*a*	21	−1.0	−1.4	4¾	110	.40	.085
4*b*	21	−1.4	−3.0	8¾		.07	.008
5	28	−0.8	−3.0	16½	1,500	.36	.022
6	66	−1.8	−1.0	35	1,500	2.46	.070

In tests Nos. 1, 2, 4, and 6 the test tubes in which the sticks were placed were immersed in the mixture of ice and water to the top of the tubes which were corked loosely to prevent interchange of air during the period of the tests. In tests Nos. 3 and 5 the sticks were placed in a short wide test tube which was securely stoppered with a rubber stopper which had passing through it a piece of narrow bore glass tubing. The test tube and stopper was completely immersed but the end of the glass tube extended above the mixture so that the pressure within the tube and around the sticks of ice was at atmospheric pressure. In test No. 5, two tubes were used, the stick of ice being placed in a smaller tube placed within the tube which was immersed in the freezing mixture. The stick, therefore, did not come into contact with the outside tube. The water formed during the test remained in the inner tube, no water being discernible on the sides of the outer tube, indicating little or no vaporization or condensation.

In none of the tests except in 1*c*, 2*g*, and 3*a*, was the temperature of the room as high as 0° C. and in these tests the final weighings were purposely delayed till the temperature was slightly above freezing in order to start the series of determinations given in Table II.

In none of the tests was the mixture of ice and water allowed to freeze solid but in all of them shells of ice were formed on the sides and bottom of the cylinder and in test 4*b* so much ice had frozen out that only about one fourth of the original amount of mixture remained liquid.

In all the tests the losses are of considerable magnitude and in some of them the heat required to melt the amount of ice melted if expressed in degrees on the amount of mixture used amounts to several tenths of a degree. For example, No. 4*a* with .4 gram of ice melted and 110 c.c. of mixture used equals 32 calories on 110 grams equal to .29 of a degree of temperature if this heat comes from the confining mixture. In 1*a* + 1*b* + 1*c* with 3.58 grams of ice melted and 1,500 c.c. of mixture, this amounts to .19 of a degree.

The tubes in which the sticks were immersed in all cases contained perceptible amounts of water and result No. 6 was in fact obtained by weighing the increase due to water rather than the loss in weight of the sticks. That heat was obtained from outside sources sufficient in amount to cause this melting is not possible as special precautions were taken to insure against such an occurrence. The mixtures used in all cases were in the room for a considerable time before using and where there seemed any possible doubt of the mixtures being at zero, they were tested by a thermometer. In no case did the temperature of the mixtures differ more than .002° from the zero point of the thermometer used. In most of the tests the freezing solution contained considerable ice crystals and in some of the tests the formation of minute ice crystals was observed upon the walls of the tube itself within a few minutes after its immersion. That the magnitude of the results obtained are different in the different tests is not surprising as differences in the size of stick and test tube used might well cause large differences in the rate of melting. Nos. 2*d*, 2*e*, 2*f*, 2*g*, and 2*h* were on the same stick as Nos. 2*a*, 2*b*, and 2*c*, but a new mixture was used, the first having frozen so nearly solid as to make a new one necessary.

In seeking an explanation, the first idea of the author was that possibly the freezing of the ice is an intermittent process and that the heat liberated by freezing and crystallization actually warms the temperature of the water to above zero. This heat is given up to any substance with which the water comes in contact and that the ice crystals themselves are not entirely melted is due to the fact that they are backed up by ice considerably below the zero temperature, hence the amounts of crystals redissolved does not equal the amount of the crystals formed and the freezing out progresses and with sufficient time the solution ultimately becomes solid. The crystals in the tube immersed in the solution, if once melted

by the absorption of any heat from the surrounding water, remain melted and the tube and stick constitute a sort of trap for absorbing heat. If, however, the melting is due to heat liberated by the freezing of ice crystals in other parts of the solution, then the melting of the sticks should cease when the mixture no longer is freezing, as would be the case when the mixture is kept in a room, the temperature of which is slightly above freezing. Table II. gives the results of a series run to test the accuracy of this hypothesis.

TABLE II.

Test No.	Approximate Weight of Stick, Gms.	Room Temp. Beginning.	Room Temp. at End.	Time in Hours.	Ice Melted in Grams.	Ice Melted per Hour in Grams.
1*d*	84	+.2	+1.4	12½	.52	.047
2*h*	34	+.2	+1.4	12¼	.51	.042
3*b*	50	+.2	+1.4	12½	.36	.029

Comparison of these results with 1*b* and 1*c*, 2*f* and 2*g* and with 3*a*, where the losses per hour are .040, .040, and .031 gram, show practically the same rate of melting as that obtained when the mixture was actually freezing in the room the temperature of which was as low as minus 3°. Evidently there must be some other explanation than the one offered.

Apparently the melting cannot be due to traces of salts in the distilled water as the outer portion of the stick was frozen first and is pure ice. Any traces of salts if present are in the interior of the stick. As the melting of the stick is on the outside the portion melted is pure ice or if not absolutely pure of the same degree of purity as that freezing out in the freezing mixture surrounding it.

A possible explanation is that the different portions of the stick are either under tension or pressure. In either case the melting point is lower than zero. Tracing the progress of the formation of a stick similar to that used in these experiments the freezing begins at the outer portion of the solution and a continuous shell of ice surrounding and enclosing a liquid interior soon forms. As freezing progresses the shell becomes thicker and thicker until the entire mass finally freezes solid. The expansion accompanying the freezing of the liquid interior exerts sufficient pressure on the outer shell of ice to cause it to burst the confining tube or to lengthen out and extend beyond this tube. The strains consequent to this stretching of the outer shell and compression of the inner portion relieve themselves partially by fracturing and cracking of the stick but the adjustment is not perfect and the frozen stick consists of an outer zone under tension and an inner zone under compression. In either case the melting point of the stick is lower than zero and the stick melts when surrounded by a mixture at this temperature.

If the lower melting point is a result of strains a release or adjustment of a relatively large portion of them is to be expected when the stick is first warmed to approximately zero and actually begins to melt. Comparison of the results of *1a*, *2a*, and *4a*, with later periods of the test, as *1b*, *2b*, etc., actually show a much greater rate during the first period. Also the results of *4b* where the rate of loss is only one tenth as great as in *4a* is in harmony with the actual conditions of the experiment as the solution surrounding the tube had frozen almost solid and was under pressure and contained any traces of salts which may have been present in the original solution. Traces of salts and pressure both lower the freezing temperature of the surrounding mixture and hence by keeping the ice in the immersed tube cooler diminishes the rate of melting or causes it to cease entirely.

If the lower melting point of the stick is a result of strains, then ice frozen under conditions where strains are not produced should remain frozen under the conditions which cause the sticks to melt. Experiments on free ice crystals obtained from the surface of distilled water freezing in an open dish and on icicles frozen from distilled water showed that both the crystals and the icicles melted when placed in tubes and the tubes immersed in the mixture of ice and water. Apparently these crystals and icicles should be at least as free from strains as the ice formed in the mixture surrounding the tubes and an explanation based on strains appears to be disproven.

A suggested explanation is that ice on first forming freezes in an unstable form which later changes into more stable crystals having a lower melting point. Buchanan[1] calls attention to the fact that lake ice during cold weather shows no crystalline figure but that on beginning to thaw a vertical prismatic structure develops. This loosening or separation of the closely crowded prisms of cold ice undoubtedly took place in the sticks used in the experiments when they were warmed up to zero. Possibly this readjustment is the reason for the lower melting point but if so it must involve some change in crystalline structure, as mere readjustment is simply an adjustment of strains, which explanation seems disproven by the melting of free ice crystals and icicles.

Sutherland[2] considers ice as $(H_2O)_3$ and water as a mixture of $(H_2O)_2$ and $(H_2O)_3$ in varying proportions and states that the true physical melting point of ice $(H_2O)_3$ is below zero, and that the true physical melting point of $(H_2O)_2$ is above zero, from which it might be inferred that the melting of ice at zero is the melting of $(H_2O)_3$, containing a

[1] Nature, Vol. 78, p. 382.

[2] Phil. Magazine, Vol. 50, pp. 460–489.

certain amount of $(H_2O)_2$. When freezing is progressing the $(H_2O)_2$ mixed with the $(H_2O)_3$ is either very small or entirely absent but when the ice warms up to approximately zero a part of the $(H_2O)_3$ is changed to $(H_2O)_2$ and the mixture begins to melt.

This possible explanation lacks experimental proof and must be taken as a possible reason rather than as an actual one.

Hess[1] in experiments on ice under different observed conditions of temperature and pressure finds as Tammann previously had observed that for a given temperature a considerable movement of the plunger occurs at a pressure lower than that figured by thermodynamical calculations, a portion of the ice melting before the total calculated pressure required to melt it was applied.

The results given in this paper may be a special case of the same phenomenon for one atmosphere of pressure and zero degrees.

SUMMARY.

The results of the experiments described establish the following fact: Dry ice sticks and crystals inside a test tube which is immersed in a mixture of ice and water will melt at the same time that other ice crystals are freezing out of the mixture surrounding the tube, the crystals inside the tube and the mixture surrounding it, both being under atmospheric pressure and the temperature of the room below being 0° C. Experimental efforts to explain the results are all negative. The suggested possible explanations lack experimental proof and are possible rather than actual reasons. The facts are accordingly presented by the author with the frank admission that he has at present no satisfactory explanation to offer for them but that he deems the facts as such to be sufficiently interesting and unexpected as to warrant their publication.

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[1] Ann. Physik., Vol. 36, pp. 449–493.

SOME EMISSION THEORIES OF LIGHT.

BY RICHARD C. TOLMAN.

THE Einstein theory of relativity assumes as its second postulate, that the velocity of light is independent of the relative motion of the source of light and the observer. It has been suggested in a number of places that all the apparent paradoxes of the Einstein theory might be avoided and at the same time the principle of the relativity of motion retained, if an alternative postulate were true that the velocity of light and the velocity of the source are additive. Relativity theories based on such a postulate may well be called emission theories.

All emission theories agree in assuming that light from a moving source has a velocity equal to the vector sum of the velocity of light from a stationary source and the velocity of the source itself at the instant of emission. Thus a source in uniform motion would always remain at the center of the spherical disturbances which it emits, these disturbances moving relative to the source itself with the same velocity c as from a stationary source.[1] Emission theories differ, however, in their assumptions as to the velocity of light after its reflection from a mirror.

If an emission theory is accepted, it would seem most natural to assume that the excited portion of a reflecting mirror acts as a new source of light and that reflected light has the same velocity c with respect to the mirror as has original light with respect to its source. The possibility of such an assumption has already been suggested by the writer[2] and apparently disproved by an experiment on the velocity of light from the approaching and receding limbs of the sun. In the present article additional evidence disproving the possibility of the assumption will be presented.

According to an emission theory suggested by Stewart[3] light reflected from a mirror acquires a component of velocity equal to the velocity of the mirror image of the original source. Evidence disproving the possibility of such a principle will also be presented in this article.

[1] Optical theories in which the velocity of light is assumed to change during the path are not considered in this article. It might be very difficult to test theories in which the velocity of light is assumed to change on passing through narrow slits or near large masses in motion, or to suffer permanent change in velocity on passing through a lense.

[2] Tolman, PHYS. REV., *31*, 26 (1910).

[3] Stewart, PHYS. REV., *32*, 418 (1911).

A very complete emission theory of electromagnetism has been presented by Ritz.[1] According to this theory light retains throughout its whole path the component of velocity which it obtained from its original moving source, and after reflection light spreads out in spherical form around a center which moves with the same velocity as the original source. In this article an experiment will be suggested whose performance would permit a decision between the Ritz and Einstein theories of relativity.

The First Emission Theory.

According to the first of the above emission theories, if a source of light is approaching an observer with the velocity v, the emitted light would have the velocity $c + v$ and after reflection from a *stationary* mirror would have the velocity c. We shall now show that measurements of the Doppler effect (in canal rays) do not agree with this theory.[2]

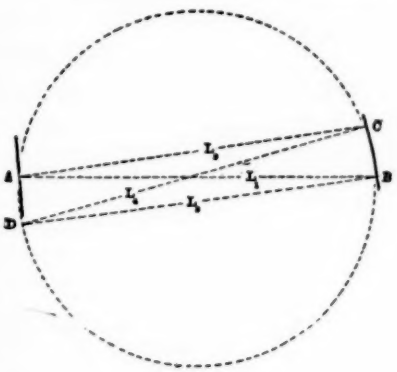

Fig. 1.

Consider measurements of the Doppler effect in light from a moving source made with a concave grating arranged as shown in Fig. 1. Light from the source (canal rays) enters the slit and falls on the grating which is so mounted that its center of curvature coincides with the position of the line of the spectrum to be photographed at D. Hence the paths BD and CD traversed after reflection by the two rays of light ABD and ACD are equal, and

[1] Ritz, Ann. de chim. et phys., *13*, 145 (1908); Arch. de Génève, *26*, 232 (1908); Scientia, *5* (1909). See also Gessamm. Werke. The Ritz electromagnetic theory does not seem to have received the critical attention which it deserves. It was the earliest systematic attempt to explain the Michelson Morley experiment on the basis of an emission theory and is the only emission theory which has been developed with any completeness.

[2] In an earlier article (loc. cit.), the author showed that if an emission theory of light were true, there would be no change in the wave-length of light when the source is set in motion. This undisputed conclusion led the author to believe that with a suitable arrangement of grating no Doppler effect would be detected in light from moving sources if an emission theory should be true. It has been correctly pointed out by Stewart (loc. cit., p. 420), however, that the use of a grating to determine wave-lengths is based on a theory which assumes a stationary medium. Hence grating measurements of the Doppler effect do not afford a general method of testing all emission theories, but such measurements must be subjected to a more complete analysis. As shown in the sequel, however, such an analysis of existing measurements of the Doppler effect is apparently sufficient to disprove the Stewart emission theory. Such measurements are not suitable for deciding between the theories of Ritz and Einstein, however, since in general these two theories would only lead to the expectation of second order differences.

the only difference in length of path occurs, before reflection, *i. e.*, $AB = L_1 > AC = L_2$.

Consider first a *stationary source*, and let τ be the period of the source which produces a bright line at D. For the production of such a line, it is evident that light impulses coming over the two paths ABD and ACD must arrive at D in the same phase. If Δt is the time interval between the departures from the source of two light impulses which arrive simultaneously at D, the condition necessary for their arrival in phase is evidently given by the equation

$$i\tau = \Delta t = \frac{L_1 - L_2}{c}, \tag{1}$$

where i is a whole number. (Note that $L_3 = L_4$ with the apparatus as arranged.)

Consider now a source of light *approaching the slit with the velocity* v. If τ' is the period of the source which now produces a bright line at D and $\Delta t'$ the time interval between departures from the source of two light impulses which now arrive simultaneously at D we evidently have the relation

$$i\tau' = \Delta t'.$$

In order to obtain an expression for $\Delta t'$ in terms of L_1 and L_2, we must note that the source moves toward the slit the distance $v\Delta t'$ during the interval of time between the departures of the two light impulses, and hence the difference in path which was $L_1 - L_2$ for a stationary source has now become $L_1 - L_2 + v\Delta t'$. Furthermore we must remember that according to the theory which we are investigating the light before reflection will have the velocity $c + v$,[1] and hence

$$i\tau' = \Delta t' = \frac{L_1 - L_2 + v\Delta t'}{c + v}, \tag{2}$$

$$i\tau' = \frac{L_1 - L_2}{c},$$

which by comparison with equation (1) gives us $\tau' = \tau$.

In other words if the first of the above emission theories of light is true, both before and after the source of light is set in motion, light produced by the same period of the source gives a bright line at the point D, that is, the expected Doppler effect or shifting of the lines does not occur.

In interpreting actual experimental results, it must be borne in mind that the adjustment of the grating was assumed to be such that the

[1] The slight difference in direction between the rays AB and AC and the motion of the source may be neglected.

reflected light is parallel to the axis of the grating. (Such an adjustment is automatically obtained with the Rowland form of mounting.) If the adjustment of the grating should be such that the difference in path all occurs after reflection it can easily be shown that the first theory *would* lead to a Doppler effect of the expected magnitude, and for intermediate adjustments to an effect of intermediate magnitude.

With regard to actual experimental results obtained with the reflected light parallel to the axis of the grating, the writer quotes from a letter received from Professor Stark.

Professor Stark says: "Sowohl in meinen Beobachtungen mit dem Konkav wie mit dem Plangitter (Ann. d. Phys., 28, 974, 1909) waren die gebeugten Strahlen, welche das beobachtete Spektrogram lieferten nicht parallel oder nahezu parallel der Gitteraxe. Doch has Paschen (Ann. d. Phys., 23, 247, 1907), soviel ich sehen kann, den Doppler Effekt bei Kanalstrahlen in der Nähe der Axe (Normalen) eines Konkavgitters beobachtet; er hat dabei mit Hilfe eines Objektivs paralleles Licht auf das Gitter fallen lassen. Ein Enterschied zwischen Paschens und meinen Reseltaten uber den Doppler-Effekt bei Kanalstrahlen hat sich indes nicht ergeben. Die zwei Methoden (einfallendes Licht parallel der Gitteraxe, gebeugtes Licht parallel dieser) liefern also bei gleicher dispersion ubereinstimmende Doppler-Effekt-Spektrogramme."

We thus see that the first of the above emission theories does not seem to accord with experimental facts.

The Stewart Theory.

By considering the same measurements of Doppler effect just described, it can also be shown that the Stewart theory does not agree with experimental facts.

Suppose a concave grating, Fig. 2, arranged as before with the center of curvature coinciding with the position of the line of the spectrum to be photographed at D.

Consider first a stationary source and let τ be the period of the source which produces a bright line at D. If Δt is the time interval between the departures from the source of two light impulses which after traveling over the two paths ABD and ACD arrive simultaneously at D, it is evident, as in the previous discussion that the condition necessary for their arrival in phase and hence for the production of a bright line is given by the equation

$$i\tau = \Delta t = \frac{L_1 - L_2}{c}, \qquad (3)$$

where i is a whole number.

Consider now a source of light approaching the slit with the velocity v. If τ' is the period of the source which now produces a bright line at D and $\Delta t'$ the time interval between departure from the source of two light

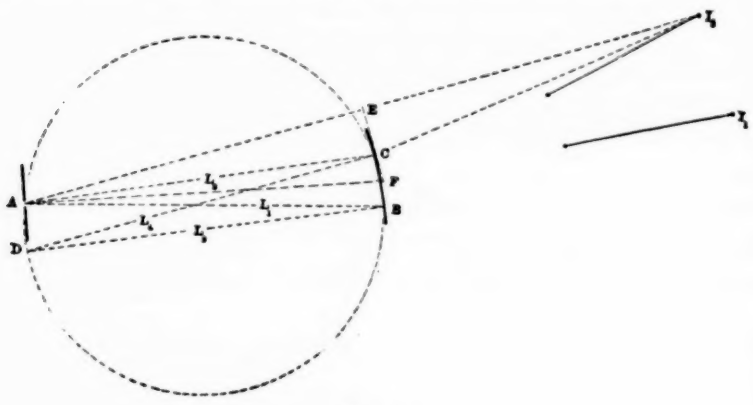

Fig. 2.

impulses which now arrive simultaneously at D, we evidently have the relation

$$i\tau' = \Delta t' = \frac{L' - L_2 + v\Delta t'}{c + v} + \frac{L_3}{c + v_3} - \frac{L_4}{c + v_4}, \qquad (4)$$

where $c + v$[1] in accordance with the Stewart theory is the velocity of the light before reflection and v_3 and v_4 are the components which must be added to c to give the velocity of light along the paths $BD = L_3$ and $CD = L_4$ after its reflection.

According to the Stewart theory v_3 and v_4 will be equal to the components in the direction BD and CD of the velocities of the mirror images of the original source. An idea of the size of these components is most easily obtained graphically. Considering, for example, the point of reflection C as a portion of a plane mirror EF which is tangent to the concave mirror at C, the position of the image I_2 can be found by the usual construction, the line AI_2 connecting source and image being perpendicular to EF and the distances AE and EI_2 equal. Both the original source and the image will evidently be moving towards the point F with the same velocity v. By a similar construction, which has been omitted to avoid confusion, the image I_1 produced by reflection from B is found to be located as shown, and moves also with the velocity v in the direction of the corresponding arrow.

It can be seen from the construction that in the arrangement shown the motion of the image I_1 and the corresponding reflected ray BD are

[1] See note 1, p. 138.

more nearly parallel than the motion of I_2 and the ray CD. Hence from the principle of Stewart the component v_3 is greater than v_4. Referring once more to equation (4), since L_3 and L_1 are equal and v_3 is greater than v_4, we see that the negative term $L_4/(c + v_4)$ is numerically greater than $L_3/(c + v_3)$ and we may write the inequality

$$\Delta t' < \frac{L_1 - L_2 + v\Delta t'}{c + v}.$$

Neglecting second order terms this becomes

$$\Delta t' \left(1 - \frac{v}{c}\right) < \frac{L_1 - L_2}{c} - \frac{L_1 - L_2}{c}\frac{v}{c}$$

and substituting from equation (3),

$$\Delta t' \left(1 - \frac{v}{c}\right) < \Delta t \left(1 - \frac{v}{c}\right),$$

$$\Delta t' < \Delta t,$$

$$\tau' < \tau.$$

Thus on the basis of the Stewart theory, with an approaching source, a shorter period would produce a bright line at the point D than with a stationary source. In other words the actual bright lines would shift towards the red end of the spectrum when the source is set in motion towards the slit, in contradiction to the actually observed shift towards the violet end of the spectrum.

We see that experimental facts do not agree with the Stewart theory.

The Ritz Theory.

According to the Ritz theory of relativity, throughout its whole path, light retains the component of velocity v which it obtained from the original moving source. Thus all the phenomena of optics would occur as though light were propagated by an ether which is stationary with respect to the original source. Light coming from a terrestial source would behave as though propagated by an ether stationary with respect to the earth and light coming from the sun would behave as though propagated by an ether stationary with respect to the sun. Now the Michelson-Morley experiment was devised for detecting the motion of the earth through the ether, and hence if this experiment should be reperformed using light from the sun instead of from a terrestrial source, a positive effect would be expected if the Ritz theory were true. On the other hand if the Einstein theory were true, no effect would be obtained, since according to this theory, all optical phenomena occur as though light were propagated by an ether stationary with respect to the observer.

To show in detail the divergence between the two theories consider the diagrammatic representation of a Michelson-Morley apparatus as shown in Fig. 3.

Light from the sun which is supposed to be moving relative to the apparatus in the direction AB with the velocity v is thrown with the help of suitable reflectors on to the half silvered mirror at A. The divided beams of light travel to the mirrors B and C and after reflection reunite at D to produce a system of interference fringes.

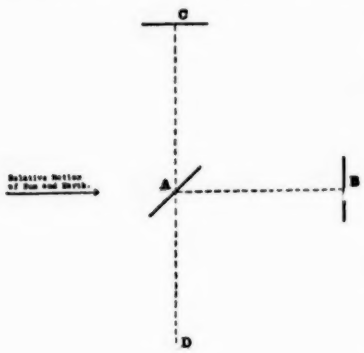

Fig. 3.

According to the Einstein theory of relativity the velocity of light is the same in all directions with respect to all observers, and hence the velocity along the paths AB and CD would be independent of the orientation of the apparatus and on the basis of this theory no change in the position of the interference fringes would be expected on rotation of the apparatus.

According to the Ritz theory, however, the velocity of light in the directions AB and AC would be different and a change in the position of the fringes would be expected on rotating the apparatus through an angle of ninety degrees.

It is easy to see that the Ritz theory would lead us to expect $c + v$ for the velocity of light in the direction AB, $c - v$ for the velocity in the opposite direction, and $\sqrt{c^2 - v^2}$ for the velocity in either direction along AC.

Assuming for simplicity that $AB = AC = l$, we see that the time required for light to travel along the path $ABBAD$ will be longer than that along the path $ACCD$ by the amount

$$\frac{l}{c+v} + \frac{l}{c-v} - \frac{2l}{\sqrt{c^2 - v^2}},$$

which neglecting terms of higher orders reduces to lv^2/c^3.

If the apparatus should be rotated through ninety degrees, it is evident that the longer time would now be required for the light to pass over the path $ACCD$ and we should expect a shift in the position of the fringes corresponding to the time interval

$$\frac{2lv^2}{c^3}.$$

Hence if the Ritz theory should be true, *using the sun as source of light* we should find on rotating the apparatus a shift in the fringes of the same magnitude as originally predicted for the Michelson-Morley apparatus where a terrestial source was used. If the Einstein theory should be true, we should find no shift in the fringes using any source of light.

SUMMARY.

Experimental evidence has been considered in this article which is apparently sufficient to disprove two of the three emission theories of light which have been proposed, and an experiment has also been suggested for testing the truth of the third emission theory, that of Ritz. A definite experimental decision between the relativity theories of Ritz and Einstein is a matter of the highest importance.

The writer wishes to express his gratitude to Dr. P. Ehrenfest for valuable suggestions and criticisms, and to Professor Stark for information concerning the adjustment of his gratings in the measurement of the Stark effect in canal rays.

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THE ELECTRICAL CONDUCTIVITY OF BISMUTH-THALLIUM ALLOYS.

BY A. E. WHITFORD.

THE recent work done by Haken[1] in investigating the electrical properties of certain alloys, especially of bismuth, has stimulated interest in this line of research. This was followed by the work of Mendenhall and Lent,[2] who determined the magnetic susceptibility of the bismuth-tellurium and bismuth-thallium series of alloys. These investigations have confirmed in the main the idea that the maxima of the melting-point curves are evidence of chemical combination, for at these points Haken found marked discontinuities in the thermo-electric curve and the electrical conductivity curve.

The present investigation was undertaken to study further the extent to which compounds as determined from the melting-point curve influence the trend of other physical properties of alloys. The specimens were cast in glass tubing in an atmosphere of hydrogen and when finished were from one to three centimeters long and about two millimeters in diameter. The metals to be fused were placed in the bottom of a small test tube which was supported vertically in an electric furnace. Two glass tubes were inserted in the cork stopper of the test tube. Through one tube the hydrogen entered the test tube and through the other, to which was attached a three-way stopcock, the hydrogen escaped to the air. The second tube however served a double purpose. It was adjustable so that it could be raised or lowered in the test tube, and by turning the three-way stopcock, an aspirator served to draw the fused metal into the glass tube. To insure a homogeneous mixture in the preparation of the alloy, in many cases the metals were kept in a molten condition in the bottom of the test tube from fifteen to thirty minutes and the test tube was frequently shaken. Especially was great care used in this respect for those mixtures whose composition was at or near the compound Bi_5Tl_3. When a cast was to be made the adjustable glass tube was lowered into the molten metal and the aspirator was started up very slowly. As soon as about 3 cm. of the metal were drawn up in the glass

[1] Haken, Ber. d. Deutsch. Gesell., March 15, 1910; Annalen der Physik, 32, 291, 1910.
[2] PHYS. REV., XXXII., No. 4, April, 1911.

tube, the stopcock was given a quarter turn, the test tube was lifted out of the furnace and the specimen thus formed was allowed to cool in the air. Inasmuch as thallium contracts somewhat on cooling, all specimens except those of pure bismuth were forced out of the glass tubes in which they were cast, by simply pushing at one end with a small tool. In the case of 100 per cent. bismuth, the glass was removed by hydrofluoric acid.

The bismuth used was Kahlbaum's chemically pure; the thallium was Schuckhardt's chemically pure. Since the composition calculated from the weights originally put into the alloys are known to check very closely with chemical analyses (Monkmeyer and Chikashige),[1] the alloys were not analyzed after the measurement of their conductivities.

The resistance of each specimen was determined by a Leeds and Northrup potentiometer. The specimen rested on two brass knife edges which were so arranged on a block that the distance between them could be varied. Uniform pressure of the specimens on the knife edges was secured by suspending the same weight from the specimen for each determination. The resistance measured varied from 0.01 ohms to 0.001 ohms and could be determined accurately to the fifth decimal place.

The accompanying table of results shows the electrical conductivity of each specimen and its method of preparation. Figure 1 shows these results graphically. There are added to the figure the melting-point curve from Monkmeyer and Chikashige,[1] and the susceptibility curve from Mendenhall and Lent.[2]

No satisfactory explanation was found for the cause of the variations in the value of the conductivity for a given percentage of mixture. This error seems to be greatest for Bi_5Tl_3 and probably is here partially explained by the fact that these points are on the steep maximum of the curve and the assumed composition may have been slightly in error. The results given are all derived from specimens which were chilled in the air immediately after casting. Some of these were afterwards annealed, but the results did not justify any conclusion as to the cause of the errors in the conductivity. It may be noted however that these variations are no greater than those in Haken's[3] determination of the conductivity of the bismuth-tellurium alloys.

As was to be expected, at 63 per cent. Bi corresponding to Bi_5Tl_4, the electrical conductivity curve shows a discontinuity, but the curve does

[1] Zeitschrift f. anorg. Chem., 46, 415, 1905, and 51, 328, 1906.

[2] Loc. cit.

[3] Annalen der Physik, 32, 291, 1910.

TABLE.

Bismuth-Thallium Alloys.

Specimen Number.	Per Cent. Bi Weight.	Method of Preparation.	El. Cond. $\times 10^7$.
7	00.00	Cut directly from stick	529.6
18	00.00	Cut directly from stick	539.5
24	10.00	From part of No. 18 by adding Bi	259.0
25	10.00	From remainder of No. 18 by adding Bi	259.3
48	10.90	From No. 17 by adding Tl	229.8
51	10.90	No. 48 remelted	228.5
10	20.00	From No. 7 by adding Bi	159.3
11	20.00	No. 10 remelted	154.3
26	20.00	From No. 25 by adding Bi	149.8
27	20.00	No. 26 remelted	152.5
17	30.00	From No. 14 by adding Tl	104.6
28	30.00	From No. 27 by adding Tl	104.5
16	40.00	Mixing No. 28 and No. 14	101.8
29	40.00	From No. 28 by adding Bi	100.1
9	50.00	From No. 7 by adding Bi	102.1
12	50.00	From No. 9 and a fresh mixture	110.2
30	50.00	From No. 29 by adding Bi	107.5
33	50.00	From No. 16 by adding Bi	107.0
14	60.00	From No. 11 by adding Bi	128.7
31	60.00	From No. 30 by adding Bi	128.2
54	61.50	Melting together	134.5
19	62.50	Melting together	139.8
22	62.50	Melting together	142.7
32	63.00	From No. 31 by adding Bi	147.3
35	63.00	Melting together	152.5
44	63.00	Melting together	160.2
46	63.00	Melting together	152.3
47	63.00	Melting together	154.5
50	64.95	Mixing No. 44 and No. 49	159.5
49	66.00	From No. 17 by adding Bi	157.9
34	70.00	From No. 16 by adding Bi	150.5
38	70.00	From No. 37 by adding Bi	156.6
41	70.00	Melting together	151.5
8	80.00	Melting together	114.3
39	80.00	From No. 38 by adding Bi	108.1
42	80.00	From No. 41 by adding Bi	112.2
4	90.00	Melting together	99.04
5	90.00	Melting together	93.45
40	90.00	From No. 38 by adding Bi	93.60
43	90.00	From No. 42 by adding Bi	96.7
1	100.00	Cut directly from stick	85.2
2	100.00	Cut directly from stick	83.6
3	100.00	Cut directly from stick	83.35

not form an ordinary cusp at this point, since its slope is steep only on one side of the point. At the other bismuth-thallium compound (10.9 per cent. Bi) no discontinuity was detected. The characteristics of this

curve agree well with those of the magnetic susceptibility curve of the bismuth-thallium series, where a well-formed cusp appears at Bi_5Tl_3, but no discontinuity is evident at the other compound point (Fig. 1).

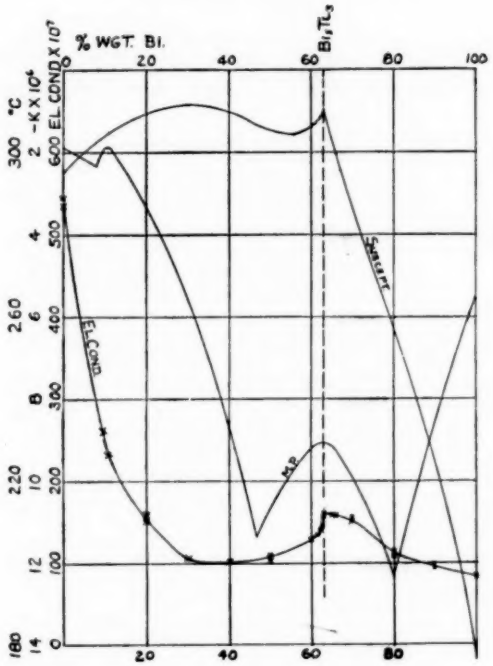

Fig. 1.

The compound containing 10.9 per cent. Bi, while it is of such a nature as to affect the melting-point curve, does not affect the conductivity or susceptibility curves.

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NOTE ON ABSOLUTE FORMULÆ FOR THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

BY G. R. OLSHAUSEN.

1. In a previous paper[1] it was shown that in the formula for the mutual inductance of two coaxial solenoids

$$M = 4\pi nn'[I_1 - I_2 - I_3 + I_4] \tag{1}$$

the terms in the brackets can be put into the form

$$I = \frac{(m\beta)^3}{2}\left[\eta_1 \wp w + \left\{\frac{g_2}{6} - \wp^2 w + \left(\frac{\wp' w}{2}\right)^2 \frac{1}{\wp w - e_3}\right\}\omega_1 + \frac{\wp' w}{2}\left\{\eta_1 w_2 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_2 i) - \tfrac{1}{2}\pi i\right\}\right]. \tag{2}$$

In this expression w_2 was determined by the relations

$$w = \omega_2 + w_2 i, \quad 0 < w_2 < \frac{\omega_3}{i},$$

w satisfying the known values of $\wp w$ and $\wp' w$ when the latter is taken with its negative sign.

In the paper cited the expression (2) had not been put into a convenient form for computation, but had only been used to derive Cohen's formula.

It is the object of this paper to adapt the expression (2) to computation and to give some additional formulæ for calculating the invariant g_2.

2. If

$$w = \omega_1 + w_1 i, \quad 0 < w_1 < \frac{\omega_3}{i}$$

is the value of w which satisfies the values of $\wp w$ and $\wp' w$ when $\wp' w$ is taken with its positive sign, then, since $\wp w$ is an even elliptic function of the second order having the periods $2\omega_1$ and $2\omega_3$, it is easy to see that

$$iw_2 = \omega_3 - iw_1. \tag{3}$$

By means of this relation and the expressions

[1] G. R. Olshausen, PHYS. REV., p. 617, Vol. 31, Dec., 1910.

$$\frac{\sigma_1'}{\sigma_1}(u-\omega_1)=\frac{\sigma'}{\sigma}(u)-\eta_1,$$

(4)

$$\frac{\sigma'}{\sigma}(u+\omega_2)=\frac{\sigma_2'}{\sigma_2}(u)+\eta_2$$

the last term of (2) can be expressed in terms of w_1.

If in the first and second equations of (4) we let

$$u=\omega_2-w_1i$$

and

$$u=-w_1i$$

respectively and then combine, we obtain

$$\frac{\sigma_1'}{\sigma_1}(w_2i)=\frac{\sigma_1'}{\sigma_1}(\omega_3-w_1i)=\eta_3-\frac{\sigma_2'}{\sigma_2}(w_1i). \tag{5}$$

Substituting (3) and (5) in the last term of (2) and noting that

$$\eta_1\omega_3-\omega_1\eta_3=\tfrac{1}{2}\pi i,$$

we get

$$I=\frac{(m\beta)^{\frac{1}{2}}}{2}\left[\eta_1\wp w+\left\{\frac{g_2}{6}-\wp^2 w+\left(\frac{\wp' w}{2}\right)^2\frac{1}{\wp w-e_3}\right\}\omega_1-\frac{\wp' w}{2}\left\{\eta_1w_1i-\omega_1\frac{\sigma_2'}{\sigma_2}(w_1i)\right\}\right], \tag{6}$$

where $\wp' w/2$ is to be taken with its negative sign.

By employing the relations existing between the σ-functions and the ϑ-functions we have, when

$$e_2<0,$$

$$\eta_1w_1i-\omega_1\frac{\sigma_2'}{\sigma_2}(w_1i)=i\pi\left\{\frac{h(z^2-z^{-2})+2h^4(z^4-z^{-4})+3h^9(z^6-z^{-6})+\cdots}{1+h(z^2+z^{-2})+h^4(z^4+z^{-4})+h^9(z^6+z^{-6})+\cdots}\right\} \tag{7}$$

and when

$$e_2>0,$$

$$\eta_1w_1i-\omega_1\frac{\sigma_2'}{\sigma_2}(w_1i)=i\pi\left\{v-\frac{i\omega_1}{\omega_3}\cdot\frac{2h_1\sin 2\pi v+4h_1^4\sin 4\pi v+6h_1^9\sin 6\pi v+\cdots}{1+2h_1\cos 2\pi v+2h_1^4\cos 4\pi v+2h_1^9\cos 6\pi v+\cdots}\right\}. \tag{8}$$

The absolute value of the expressions (7) and (8) is smaller than that of the corresponding expressions (52) and (59) of the former paper,[1] the symbols having the same significance.

3. By a proper choice of the parameter m it will always be possible to calculate accurately the value of $g_2/6$ from the expression

[1] Loc. cit.

$$\frac{g_2}{6} = \frac{2}{9m^2}\left[\left(\frac{a_1^2 + a^2 + c^2}{2a_1 a}\right)^2 + 3\right]. \tag{9}$$

This quantity may, however, also be computed by means of the ϑ-functions as follows. We have, when

$$e_2 < 0,$$

$$\begin{aligned}\frac{g_2}{6} &= \frac{1}{9}\left(\frac{\pi}{2\omega_1}\right)^4 [\vartheta_2^8(0) + \vartheta_3^8(0) + \vartheta_0^8(0)] \\ &= \frac{1}{9}\cdot\frac{1}{4^4}\left[\frac{\sqrt{e_1 - e_3} + \sqrt[4]{e_1 - e_2}}{1 + h^4 + 2h^{16} + \cdots}\right]^8 [\vartheta_2^8(0) + \vartheta_3^8(0) + \vartheta_0^8(0)],\end{aligned} \tag{10}$$

where

$$\begin{aligned}\vartheta_2(0) &= 2h^{\frac{1}{4}}(1 + h^2 + h^6 + h^{12} + \cdots), \\ \vartheta_3(0) &= 1 + 2h + 2h^4 + 2h^9 + \cdots, \\ \vartheta_0(0) &= 1 - 2h + 2h^4 - 2h^9 + \cdots.\end{aligned}$$

On substituting these expressions in (10) and expanding, we obtain

$$\begin{aligned}\frac{g_2}{6} = \frac{1}{2304}&\left[\frac{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}}{1 + 2h^4 + 2h^{16} + \cdots}\right]^8 \\ &\times [2 + 480h^2 + 4320h^4 + 13440h^6 + 35040h^8 \cdots]\end{aligned}$$

or

$$\begin{aligned}\frac{g_2}{6} = &\frac{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3})^8}{1152} \\ &\times (1 + 240h^2 + 2144h^4 + 2880h^6 - 16896h^8 \cdots).\end{aligned} \tag{11}$$

When

$$e_2 > 0,$$

$$\begin{aligned}\frac{g^2}{6} &= \frac{1}{9}\left(\frac{\pi i}{2\omega_3}\right)^4 [\vartheta_2^8(0 \mid \tau_1) + \vartheta_3^8(0 \mid \tau_1) + \vartheta_0^8(0 \mid \tau_1)] \\ &= \frac{1}{9}\cdot\frac{1}{4^4}\left[\frac{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3}}{1 + 2h_1^4 + 2h_1^{16} \cdots}\right]^8 [\vartheta_2^8(0 \mid \tau_1) + \vartheta_3^8(0 \mid \tau_1) + \vartheta_0^8(0 \mid \tau_1)],\end{aligned} \tag{12}$$

where

$$\begin{aligned}\vartheta_2(0 \mid \tau_1) &= 2h_1^{\frac{1}{4}}(1 + h_1^2 + h_1^6 + h_1^{12} + \cdots), \\ \vartheta_3(0 \mid \tau_1) &= 1 + 2h_1 + 2h_1^4 + 2h_1^9 + \cdots, \\ \vartheta_0(0 \mid \tau_1) &= 1 - 2h_1 + 2h_1^4 - 2h_1^9 + \cdots.\end{aligned}$$

On substituting these values and expanding, we obtain in the same manner as before

$$\begin{aligned}\frac{g_2}{6} = &\frac{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3})^8}{1152} \\ &\times (1 + 240h_1^2 + 2144h_1^4 + 2880h_1^6 - 16896h_1^8 \cdots).\end{aligned} \tag{13}$$

It is to be noted that when $e_2 \leqq 0$, $h \leqq e^{-\pi} < 1/23$ and when $e_2 \geqq 0$, $h_1 \leqq e^{-\pi} < 1/23$, so that the terms $16896h^8$ and $16896h_1^8$, are then both less than 2×10^{-7}.

4. Nagaoka[1] has also employed the ϑ-functions to determine $g_2/6$, but his formulæ do not converge very rapidly when $\sqrt{k'}$ is small. This occurs when in the expression

$$k'^2 = \frac{(a_1 - a)^2 + c^2}{(a_1 + a)^2 + c^2}$$

the numerator is small compared to $(a_1 + a)^2$, for example, when two nearly concentric coils of the same length have nearly the same radius.

For the case computed in the paper referred to above, where

$$d = 0,\ 2l = 200,\ 2l_1 = 20,\ a = 10,\ a_1 = 15,$$

the results of the computation are given below.

In this computation $g_2/6$ was calculated by equation (9), the parameter m having been so chosen as to make $e_1 - e_3 = 1$.

The quantities

$$l = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \text{ and } lt' = -\frac{1 - \sqrt{\frac{e_1 - s}{k'}}}{1 + \sqrt{\frac{e_1 - s}{k'}}}$$

were computed by means of the relation

$$\frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}$$

by letting $\cos x$ equal $\sqrt{k'}$ and $\sqrt{\frac{e_1 - s}{k'}}$ respectively.

If equation (11) is used to compute $g_2/6$, we find

$$\frac{g_2}{6} = 0.21223833 \text{ for } c = 110,$$

$$\frac{g_2}{6} = 0.20799141 \text{ for } c = 90$$

and

$$M = 4\pi nn' 6213.501$$

instead of

$$M = 4\pi nn' 6213.428$$

as given below.

The values of $g_2/6$ found by using (9) are the better ones, however, and were checked by means of Crelle's multiplication tables.

On recalculating M by (21) and (61)[2] with greater precision than before,

[1] H. Nagaoka, Jour. Coll. Sci., Tokyo, 27, art. 6, 1909.

[2] G. R. Olshausen, l. c.

we find

$$M = 4\pi nn' 6213.482$$

instead of

$$M = 4\pi nn' 6213.77.$$

	$c = 110.$	$c = 90.$
$\log k'^2 = \log \frac{(a_1 - a)^2 + c^2}{(a_1 + a)^2 + c^2}$	9.9790239	9.9690580
$\log \sqrt{k'} = \log \cos x$	9.9947560	9.9922645
$\log l = \log \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \log \tan^2 \frac{x}{2}$	7.7808467	7.9496635
$\log h = \log \left(\frac{l}{2} + \cdots \right)$	7.4798167	7.6486335
$\log \omega_1 = \log \frac{2\pi}{(1 + \sqrt{k'})^2} (1 + \cdots)^2$	0.2013480	0.2038209
$\log \eta_1 = \log \frac{\pi^2}{12\omega_1} \frac{1 - 27h^2 \cdots}{1 - 3h^2 \cdots}$	9.7136755	9.7110910
$\log \mathfrak{p}w = \log \frac{1}{3} \frac{2(a_1^2 + a^2) - c^2}{(a_1 + a)^2 + c^2}$	9_n.4770264	9_n.4542696
$\log \sqrt{e_1 - s} = \log \sqrt{\frac{c^2}{(a_1 + a)^2 + c^2}}$	9.9890638	9.9838598
$\log \sqrt{\frac{e_1 - s}{k'}} = \log \cos x'$	9.9943078	9.9915953
$\log ll' = \log \tan^2 \frac{x'}{2}$	7_n.8164594	7_n.9856954
$- l'$	1.0854573	1.0865056
$\log hz^2 = \log (- \sqrt{l'^2 - 1} - l')$	9.8217095	9.8206346
$\log \left\{ \frac{h(z^2 - z^{-2}) + \cdots}{1 + h(z^2 + z^{-2}) + \cdots} \right\}$	9.6007265	9.6000650
$\log \frac{\mathfrak{p}'w}{2i} = \log \frac{(a_1^2 + a^2)c}{[(a_1 + a)^2 + c^2]^{\frac{3}{2}}}$	7_n.9813160	8_n.1400043
$\log (\mathfrak{p}w - e_3) = \log \frac{(a_1 + a)^2}{(a_1 + a)^2 + c^2}$	8.6912222	8.8551146
$\log \frac{\mathfrak{p}'w}{2} \left\{ \eta_1 w_1 i - \omega_1 \frac{\sigma_3'}{\sigma_3} (w_1 i) \right\}$	8.0791924	8.2372192
$\frac{g_2}{6} = \frac{2}{9} \left[\left(\frac{a_1^2 + a^2 + c^2}{(a_1 + a)^2 + c^2} \right)^2 + 3 \left(\frac{2a_1 a}{(a_1 + a)^2 + c^2} \right)^2 \right]$	0.21223822	0.20799139
$\mathfrak{p}^2 w$	0.08996069	0.08101011
$\left(\frac{\mathfrak{p}'w}{2} \right)^2 \frac{1}{\mathfrak{p}w - e_3}$	−0.00186814	−0.00266012
$\lambda = \omega_1 \left\{ \frac{g_2}{6} - \mathfrak{p}^2 w + \left(\frac{\mathfrak{p}'w}{2} \right)^2 \frac{1}{\mathfrak{p}w - e_3} \right\}$	0.19142930	0.19877689
$\mu = - \frac{\mathfrak{p}'w}{2} \left\{ \eta_1 w_1 i - \omega_1 \frac{\sigma_2'}{\sigma_2} (w_1 i) \right\}$	−0.01200031	−0.01726709
$\nu = \mathfrak{p}w \cdot \eta_1$	−0.15513219	−0.14633915
$\lambda + \mu + \nu$	0.02429681	0.03517065
$\log (\lambda + \mu + \nu)$	8.3855510	8.5461804
$\log (m\beta)^{\frac{3}{2}} = \log [(a_1 + a)^2 + c^2]^{\frac{3}{2}}$	6.1569867	5.9111482
$\log 2I = \log (m\beta)^{\frac{3}{2}} (\lambda + \mu + \nu)$	4.5425377	4.4573286
$2I$	34876.887	28663.459

$$2(I_1 - I_2) = 2(I_4 - I_3) = 34876.887 - 28663.459 = 6213.428,$$
$$M = 4\pi nn' 6213.428.$$